STRONG LAW OF LARGE NUMBERS FOR LEVEL-WISE INDEPENDENT FUZZY RANDOM VARIABLES

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ABSTRACT. In this paper, we obtain a strong law of large numbers for sums of level-wise independent and level-wise identically distributed fuzzy random variables.

1. Introduction

Laws of large numbers for sums of independent random sets have been studied by Artstein and Hart [1], Artstein and Vitale [2], Puri and Ralescu [16], Taylor and Inoue [18], Uemura [19], etc. These results have been generalized to the case of fuzzy random variables by several people. A SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Kruse [14], and a SLLN for sums of independent fuzzy random variables by Miyakoshi and Shimbo [15]. Also, Klement, Puri and Ralescu [12] proved some limit theorems which includes a SLLN, and Inoue [10] obtained a SLLN for sums of independent tight fuzzy random sets. Recently, Hong and Kim [9] generalized Marcinkiewicz law of large numbers to fuzzy random variables.

In this paper, we obtain a SLLN for sums of level-wise independent and level-wise identically distributed fuzzy random variables by using a metric which is stronger than one in works mentioned previously. The representation theorem of fuzzy numbers by Goetschel and Voxman [7] will be used.

2. Preliminaries

In this section, we describe some basic concepts of fuzzy numbers. Let $R$ denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \to [0, 1]$ with the
following properties;

(1) \( \tilde{u} \) is normal, i.e., there exists \( x \in \mathbb{R} \) such that \( \tilde{u}(x) = 1 \).

(2) \( \tilde{u} \) is upper semicontinuous.

(3) \( \text{supp } \tilde{u} = \text{cl}\{x \in \mathbb{R} : \tilde{u}(x) > 0\} \) is compact.

(4) \( \tilde{u} \) is a convex fuzzy set, i.e., \( \tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y)) \) for \( x, y \in \mathbb{R} \) and \( \lambda \in [0, 1] \).

For a fuzzy set \( \tilde{u} \), we define

\[
L_{\alpha}\tilde{u} = \begin{cases} 
\{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha < 1 \\
\text{supp } \tilde{u}, & \alpha = 0
\end{cases}
\]

Then, it follows that \( \tilde{u} \) is a fuzzy number if and only if \( L_1\tilde{u} \neq \phi \) and \( L_{\alpha}\tilde{u} \) is a closed bounded interval for each \( \alpha \in [0, 1] \). From this characterization of fuzzy numbers, a fuzzy number \( \tilde{u} \) is completely determined by the end points of the intervals \( L_{\alpha}\tilde{u} = [u^-_{\alpha}, u^+_{\alpha}] \). We denote the family of all fuzzy numbers by \( F(\mathbb{R}) \).

**Theorem 2.1 ([7]).** For \( \tilde{u} \in F(\mathbb{R}) \), we denote \( u^-(\alpha) = u^-_{\alpha} \) and \( u^+(\alpha) = u^+_{\alpha} \). Then the followings hold;

(1) \( u^-(\alpha) \) is a bounded increasing function on \([0,1]\).

(2) \( u^+(\alpha) \) is a bounded decreasing function on \([0,1]\).

(3) \( u^-(1) \leq u^+(1) \).

(4) \( u^-(\alpha) \) and \( u^+(\alpha) \) are left continuous on \((0,1]\) and right continuous at \(0\).

(5) If \( \tilde{v} \) is a fuzzy number satisfying above (1)-(4), then there exists unique \( \tilde{v} \in F(\mathbb{R}) \) such that \( \tilde{v}^-_{\alpha} = v^-(\alpha) \), \( \tilde{v}^+_{\alpha} = v^+(\alpha) \).

The above theorem implies that we can identify a fuzzy number \( \tilde{u} \) with the parametrized representation \( \{(u^-_{\alpha}, u^+_{\alpha})| 0 \leq \alpha \leq 1\} \). Suppose now that \( \tilde{u}, \tilde{v} \) are fuzzy numbers represented by \( \{(u^-_{\alpha}, u^+_{\alpha})| 0 \leq \alpha \leq 1\} \) and \( \{(v^-_{\alpha}, v^+_{\alpha})| 0 \leq \alpha \leq 1\} \), respectively. If we define

\[
(\tilde{u} + \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),
\]

\[
(\lambda \tilde{u})(z) = \begin{cases} 
\tilde{u}(z/\lambda), & \lambda \neq 0 \\
\tilde{0}, & \lambda = 0
\end{cases}
\]
where \( \tilde{0} = I_{\{0\}} \) is the indicator function of \( \{0\} \), then
\[
\tilde{u} + \tilde{v} = \{(u_\alpha^- + v_\alpha^-, v_\alpha^+ + v_\alpha^+) | 0 \leq \alpha \leq 1\},
\]
\[
\lambda \tilde{u} = \begin{cases} 
\{(\lambda u_\alpha^-, \lambda u_\alpha^+) | 0 \leq \alpha \leq 1\}, & \lambda \geq 0 \\
\{(\lambda u_\alpha^+, \lambda u_\alpha^-) | 0 \leq \alpha \leq 1\}, & \lambda < 0.
\end{cases}
\]

Now, we define two metrics \( d, d^* \) on \( F(R) \) by
\begin{align}
(2.1) \quad d(\tilde{u}, \tilde{v}) &= \sup_{0 \leq \alpha \leq 1} d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v}) \\
(2.2) \quad d^*(\tilde{u}, \tilde{v}) &= \int_0^1 d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v}) d\alpha
\end{align}
where \( d_H \) is the Hausdorff metric defined as
\[
d_H(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|).
\]
Also, the norm \( \| \tilde{u} \| \) of fuzzy number \( \tilde{u} \) will be defined as
\[
\| \tilde{u} \| = d(\tilde{u}, \tilde{0}) = \max(|u_0^-|, |u_0^+|).
\]

3. Fuzzy random variables

Throughout this paper, \((\Omega, A, P)\) denotes a complete probability space. If \( \tilde{X} : \Omega \to F(R) \) is a fuzzy number valued function and \( B \) is a subset of \( R \), then \( \tilde{X}^{-1}(B) \) denotes the fuzzy subset of \( \Omega \) defined by
\[
\tilde{X}^{-1}(B)(\omega) = \sup_{x \in B} \tilde{X}(\omega)(x)
\]
for every \( \omega \in \Omega \). The function \( \tilde{X} : \Omega \to F(R) \) is called a fuzzy random variable if for every closed subset \( B \) of \( R \), the fuzzy set \( \tilde{X}^{-1}(B) \) is measurable when considered as a function from \( \Omega \) to \([0, 1] \). If we denote \( \tilde{X}(\omega) = \{(X_\alpha^- (\omega), X_\alpha^+ (\omega)) | 0 \leq \alpha \leq 1\} \), then it is well-known that \( \tilde{X} \) is a fuzzy random variable if and only if for each \( \alpha \in [0, 1] \), \( X_\alpha^- \) and \( X_\alpha^+ \) are random variables in the usual sense (See Kim and Ghil [11]). Hence, if \( \sigma(\tilde{X}) \) is the smallest \( \sigma \)-field which makes \( \tilde{X} \) a fuzzy random variable, then \( \sigma(\tilde{X}) \) is consistent with \( \sigma(\{X_\alpha^-, X_\alpha^+ | 0 \leq \alpha \leq 1\}) \). This enables us to define the concept of independence for fuzzy random variables as in the case of classical random variables.
Definition 3.1. Let \( \tilde{X}, \tilde{Y} \) be two fuzzy random variables whose representations are \( \{(X^-_\alpha, X^+_\alpha) | 0 \leq \alpha \leq 1\} \) and \( \{(Y^-_\alpha, Y^+_\alpha) | 0 \leq \alpha \leq 1\} \), respectively.

1. \( \tilde{X} \) and \( \tilde{Y} \) are called independent if the \( \sigma \)-fields \( \sigma(\tilde{X}) \) and \( \sigma(\tilde{Y}) \) are independent.

2. \( \tilde{X} \) and \( \tilde{Y} \) are called level-wise independent if for each \( \alpha \in [0, 1] \) the \( \sigma \)-fields \( \sigma(X^-_\alpha, X^+_\alpha) \) and \( \sigma(Y^-_\alpha, Y^+_\alpha) \) are independent.

3. \( \tilde{X} \) and \( \tilde{Y} \) are called level-wise identically distributed if for each \( \alpha \in [0, 1] \), \( (X^-_\alpha, X^+_\alpha) \) and \( (Y^-_\alpha, Y^+_\alpha) \) are identically distributed random vectors.

Note that the definitions (2) and (3) is firstly introduced in this paper.

Definition 3.2. A fuzzy random variable \( \tilde{X} = \{(X^-_\alpha, X^+_\alpha) | 0 \leq \alpha \leq 1\} \) is called integrable if for each \( \alpha \in [0, 1] \), \( X^-_\alpha \) and \( X^+_\alpha \) are integrable, equivalently, \( \int \|\tilde{X}\| dP < \infty \). In this case, the expectation of \( \tilde{X} \) is defined by

\[
E \tilde{X} = \int \tilde{X} dP = \{\left(\int X^-_\alpha dP, \int X^+_\alpha dP\right) | 0 \leq \alpha \leq 1\}
\]

4. Main Result

In this section, a SLLN with respect to the metric \( d \) defined as in (2.1) will be obtained. In earlier works, the metric \( d^* \) defined as in (2.2) have been used (see [9], [10], [12]). Note that \( d \) is stronger than \( d^* \). First, we need a subspace \( F_C(R) \) of \( F(R) \). Let \( F_C(R) = \{u \in F(R) | u^-_\alpha \) and \( u^+_\alpha \) are continuous when considered as functions of \( \alpha \}\}.

Then it is known that \( \tilde{u} \in F_C(R) \) if and only if for any \( \beta \in (0, 1) \), there exist at most two different \( x_1, x_2 \) such that \( \tilde{u}(x_1) = \tilde{u}(x_2) = \beta \) (See [4] Theorem 5.1). Note that if \( \tilde{X} \) is \( F_C(R) \)-valued, then \( E \tilde{X} \in F_C(R) \).

Before we state the main result, we recall the following lemma which is well-known in the classical Analysis.

Lemma 4.1. Let \( (f_n) \) be a sequence of monotonic functions on \( [0, 1] \). If \( f_n(x) \) converges pointwise to a continuous function \( f(x) \) on \( [0, 1] \), then \( f_n(x) \) converges to \( f(x) \) uniformly.

We now state the SLLN for sums of level-wise independent fuzzy random variables.
THEOREM 4.2. Let \( \{\tilde{X}_n\} \) be a sequence of level-wise independent and level-wise identically distributed fuzzy random variables with \( E\|\tilde{X}_1\| < \infty \). If \( E\tilde{X}_1 \in F_C(R) \), then

\[
\lim_{n \to \infty} d\left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1 \right) \to 0 \text{ a.s.}
\]

PROOF. Let \( \tilde{X}_n = \{(X_{na}^-, X_{na}^+) | 0 \leq \alpha \leq 1\} \). Then for each \( \alpha \in [0, 1] \), \( \{(X_{na}^-, X_{na}^+)\} \) is a sequence of independent and identically distributed random vectors with \( E|X_{na}^-| < \infty \) and \( E|X_{na}^+| < \infty \) in the classical sense. By Kolmogorov’s strong law of large numbers,

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- \to EX_{1\alpha}^- \text{ a.s.}
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^+ \to EX_{1\alpha}^+ \text{ a.s.}
\]

Now, let \( \{r_k\} \) be a countable dense subset of \([0, 1]\) with \( r_0 = 0, r_1 = 1 \). Then there exist \( B_k \in \mathcal{A} \) with \( P(B_k) = 0 \) such that for each \( \omega \notin B_k \),

(4.1) \[
\frac{1}{n} \sum_{i=1}^{n} X_{ir_k}^- (\omega) \to EX_{1r_k}^-
\]

(4.2) \[
\frac{1}{n} \sum_{i=1}^{n} X_{ir_k}^+ (\omega) \to EX_{1r_k}^+
\]

If we define \( B = \bigcup_{k=0}^{\infty} B_k \), then \( P(B) = 0 \) and for each \( \omega \notin B \), (4.1) and (4.2) hold for all \( r_k \). Now, we will show that for each \( \omega \notin B \),

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) \to EX_{1\alpha}^- \text{ uniformly in } \alpha \in [0, 1].
\]
By Lemma 4.1, it suffices to show that for each $\omega \notin B$, and each $\alpha$,

$$\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) \to E X_{1\alpha}^-.$$

Let $\omega \notin B$ and $\epsilon > 0$ be fixed. Then by the continuity of $E X_{1\alpha}^-$ as a function of $\alpha$, there exists $\delta > 0$ such that

$$|\alpha - \beta| < \delta \implies |E X_{1\alpha}^- - E X_{1\beta}^-| < \epsilon$$

If we take $r_l, r_m$ so that $\alpha - \delta < r_l < \alpha < r_m < \alpha + \delta$, then

$$E X_{1r_{m}}^- - \epsilon < E X_{1\alpha}^- < E X_{1r_{l}}^- + \epsilon.$$

Hence, by the monotonicity of $X_{i\alpha}^- (\omega)$ with respect to $\alpha$,

$$\frac{1}{n} \sum_{i=1}^{n} X_{ir_{l}}^- (\omega) - E X_{1r_{l}}^- - \epsilon < \frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) - E X_{1\alpha}^-$$

$$< \frac{1}{n} \sum_{i=1}^{n} X_{ir_{m}}^- (\omega) - E X_{1r_{m}}^- + \epsilon$$

which implies

$$\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- (\omega) \to E X_{1\alpha}^-.$$

Similarly, it can be proved that for each $\omega \notin B$

$$\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^+ (\omega) \to E X_{1\alpha}^+ \text{ uniformly in } \alpha \in [0, 1].$$

Therefore, for each $\omega \notin B$.

$$d \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_{i}(\omega), E \tilde{X}_1 \right) \to 0.$$
COROLLARY 4.3. Let \( \{\bar{X}_n\} \) be a sequence of level-wise independent and level-wise identically distributed \( F_C(R) \)-valued fuzzy random variables. There exists \( \bar{b} \in F_C(R) \) such that

\[
(4.3) \quad d\left( \frac{1}{n} \sum_{i=1}^{n} \bar{X}_i, \bar{b} \right) \longrightarrow 0 \quad a.s.
\]

if and only if \( E\|\bar{X}_1\| < \infty \). Furthermore, if (4.3) holds, then \( \bar{b} = E\bar{X}_1 \).

Proof. The sufficiency follows immediately from theorem 4.2. To prove the converse, if (4.3) holds, then for any \( \alpha \in [0, 1] \),

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^- \longrightarrow b^-_\alpha \quad a.s.
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} X_{i\alpha}^+ \longrightarrow b^+_\alpha \quad a.s.
\]

By the converse of Kolmogorov’s strong law of large numbers,

\[
E |X_{1\alpha}^-| < \infty, E |X_{1\alpha}^+| < \infty \quad \text{for each } \alpha \in [0, 1]
\]

which implies \( E\|\bar{X}_1\| < \infty \) and \( \bar{b} = E\bar{X}_1 \).

Example. Let \( \bar{u} \in F_c(R) \) be fixed and let \( \{Y_n\} \) be i.i.d. with \( E|Y_1| < \infty \) in the usual sense. Define \( \bar{X}_n(w)(x) = \bar{u}(x - Y_n(w)) \) i.e., \( \bar{X}_n(w) \) is the translation of \( \bar{u} \) by \( Y_n(w) \) in \( x \)-axis. Then

\[
X_{n,\alpha}^-(w) = u^-_\alpha + Y_n(w) \quad \text{and} \quad X_{n,\alpha}^+(w) = u^+_\alpha + Y_n(w)
\]

Hence the above theorem implies that

\[
d\left( \frac{1}{n} \sum_{i=1}^{n} \bar{X}_i, E\bar{X}_1 \right) \longrightarrow 0 \quad a.s.
\]

where \( (E\bar{X}_1)(x) = \bar{u}(x - EY_1) \).

As a final result, we give a generalization of Chung’s SLLN to the case of fuzzy random variables.
THEOREM 4.4. Let \( \{\tilde{X}_n\} \) be a sequence of fuzzy random variables. If \( \{\|\tilde{X}_n\|\} \) are independent random variables in classical sense and

\[
\sum_{n=1}^{\infty} \frac{1}{n} E\|\tilde{X}_n\| < \infty, \tag{4.4}
\]

then

\[
d\left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n} \sum_{i=1}^{n} E\tilde{X}_i \right) \to 0 \ a.s.
\]

PROOF. First we note that

\[
d\left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, \frac{1}{n} \sum_{i=1}^{n} E\tilde{X}_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} d(\tilde{X}_i, E\tilde{X}_i) \\
\leq \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\| + E\|\tilde{X}_i\|).
\]

Since \( \{\|\tilde{X}_n\|\} \) is a sequence of independent random variables, (4.4) and Chung’s law of large numbers yields

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\| - E\|\tilde{X}_i\|) = 0 \ a.s. \tag{4.5}
\]

Now, applying the Kronecker lemma to (4.4), we obtain

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E\|\tilde{X}_i\| = 0
\]

which implies, together with (4.5),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\|\tilde{X}_i\|) = 0 \ a.s.
\]

This gives the desired result. \( \square \)
References

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