SOME FIXED POINT THEOREMS FOR
CONTRACTIVE AND EXPANSIVE MAPS

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Abstract. In this paper, fixed point theorems for contractive and
expansive maps are established, some of which extend a few results
of Das and Debata, Edelstein, Fisher, Leader, Shih and Yeh, and
Jungck.

1. Introduction

Let \( f \) and \( g \) be continuous self maps of a compact metric space \((X, d)\)
and let \( N \) be the set of positive integers. For \( x, y \in X \) and \( A, B \subset X \),
define

\[
O(x, f) = \{ f^n x \mid n \in N \cup \{0\} \}, \\
O(x, y, f) = O(x, f) \cup O(y, f), \\
O(x, y, f, g) = O(x, y, f) \cup O(x, y, g), \\
\delta(A, B) = \sup \{ d(a, b) \mid a \in A, b \in B \}.
\]

Let \( \delta(A) \) denote the diameter of \( A \). Define

\[
C_f = \{ h \mid h : X \to X \text{ and } hf = fh \}, \\
A_f = \{ h \mid h : X \to X \text{ and } h \cap_{n \in N} f^n X = \cap_{n \in N} f^n X \}, \\
H_f = \{ h \mid h : X \to X \text{ and } h \cap_{n \in N} f^n X \subset \cap_{n \in N} f^n X \}.
\]

Clearly \( C_f, A_f \) and \( H_f \) are semigroups under composition. Let \( \mathcal{F} \)
and \( \mathcal{T} \) be families of self maps on \( X \). A point \( x \) in \( X \) is called a fixed
point of \( \mathcal{F} \) if \( fx = x \) for \( f \in \mathcal{F} \), a common fixed point of \( \mathcal{F} \) and \( \mathcal{T} \) if
\( fx = gx = x \) for \( f \in \mathcal{F} \) and \( g \in \mathcal{T} \).

Edelstein [2] established the existence of a unique fixed point of a self
map \( f \) of a compact metric space satisfying the inequality \( d(fx, fy) < \)

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\[ d(x, y) \]. Das and Debata [1], Fisher [3], Leader [7], Shih and Yeh [8] obtained a number of generalizations of this result. Jungck [6] proved two interesting results on fixed point in compact metric spaces, one of which deals with the existence of fixed point of \( C_{gf} \) and extends the results of Das and Debata [1], Edelstein [2], Fisher [3], Leader [7], Shih and Yeh [8].

The main purpose of this paper is to extend Jungck’s results to a few much wider classes of maps. In section 2, fixed point theorems are proved by considering a few contractive types conditions for \( H_{gf} \), \( H_f \) and \( H_g \). In section 3, fixed point theorems are established by considering a few expansive types conditions for \( H_{gf} \), \( C_f \) and \( C_g \).

By Proposition 4.1 of Jungck [6] and Proposition 1 of Leader [7], we obtain the following lemmas:

**Lemma 1.1.** Let \( f \) be a continuous self map of a compact metric space \( (X, d) \). Let \( A = \cap_{n \in \mathbb{N}} f^n X \). Then

(i) \( A \) is a nonempty compact subset of \( X \);

(ii) \( \{f^n | n \in \mathbb{N} \cup \{0\}\} \subset A_f \cap C_f \);

(iii) \( C_f \cup A_f \subset H_f \).

**Lemma 1.2.** Let \( f \) and \( g \) be self maps of a compact metric space \( (X, d) \) such that \( gf \) is continuous and \( f \in A_{gf} \). Then \( g \in A_{gf} \).

**Lemma 1.3.** Let \( f \) and \( g \) be commuting self maps of a compact metric space \( (X, d) \) such that \( gf \) is continuous. Then \( f, g \in A_{gf} \).

### 2. Fixed point theorems for \( H_{gf} \), \( H_f \) and \( H_g \)

**Theorem 2.1.** Let \( f \) and \( g \) be self maps of a compact metric space \( (X, d) \) such that \( gf \) is continuous and \( f \in A_{gf} \). Assume that there exist \( S, T \in A_{gf} \) satisfying

\[
(2.1) \quad d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g))
\]

for \( Sx \neq Ty \). Then \( f, g, S \) and \( T \) have a unique common fixed point which is a unique fixed point of \( H_{gf} \).

**Proof.** Let \( A = \cap_{n \in \mathbb{N}} (gf)^n X \). It follows from (i) of Lemma 1.1 that \( A \) is a nonempty compact subset of \( X \). Thus there exist \( a, b \in A \) such that \( \delta(A) = d(a, b) \). Since \( S, T \in A_{gf} \), we can find \( x, y \in A \) such that \( Sx = a \) and \( Ty = b \). By Lemma 1.2 we have \( g \in A_{gf} \). Note that
$f \in A_{gf}$. Then $O(x, y, f, g) \subset A$. We assert that $A$ is a singleton. If not, then $\delta(A) > 0$. Using (2.1),

$$d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g))$$
$$\leq \delta(\cup_{h \in H_{gf}} h A)$$
$$\leq \delta(A),$$

which implies that

$$0 < \delta(A) = d(Sx, Ty) < \delta(A),$$

which is impossible. Hence $A$ is a singleton, i.e., $A = \{w\}$ for some $w$ in $X$. This implies that $w$ is a fixed point of $H_{gf}$, in particular, $w$ is a common fixed point of $f, g, S$ and $T$.

If $v$ is another common fixed point of $f, g, S$ and $T$, then $(gf)^n = v$ for all $n$ in $N$. This implies $v \in A$ and $v = w$. Hence $f, g, S$ and $T$ have a unique common fixed point $w$. Note that $f, g, S$ and $T \in A_{gf} \subset H_{gf}$. Therefore $H_{gf}$ has a unique fixed point $w$. This completes the proof. $\Box$

**Corollary 2.1.** Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous and $f \in A_{gf}$. If $fx \neq gy$ implies

$$d(fx, gy) < \delta(\cup_{h \in H_{gf}} h O(x, y, f, g)),$$

then $f$ and $g$ have a unique common fixed point which is a unique fixed point of $H_{gf}$.

**Proof.** Corollary 2.1 follows from Lemma 1.2 and Theorem 2.1. $\Box$

**Remark 2.1.** By (iii) of Lemma 1.1 and Lemma 1.3 and Example 3.1 in section 3, it follows that Corollary 2.1 generalizes properly Theorem 4.2 of Jungck [6].

**Corollary 2.2.** Let $f$ be a continuous self map of a compact metric space $(X, d)$. Assume that there exist $S, T \in A_f$ satisfying

$$d(Sx, Ty) < \delta(\cup_{h \in H_f} h O(x, y, f))$$

for $Sx \neq Ty$. Then $f$ has a uniformly contractive fixed point which is a unique fixed point of $H_f$. 

Proof. Take $g = i_X$ (the identity map) in Theorem 2.1. By (ii) of Lemma 1.1, $f \in A_f$. Note that $O(x, y, f, i_X) = O(x, y, f)$. It follows from Theorem 2.1 that $\cap_{n \in N} f^n X = \{w\}$ and $w$ is a unique fixed point of $H_f$. By Theorem 1 of Leader [7], we conclude that $f$ has a uniformly contractive fixed point $w$. This completes the proof. \[\square\]

Remark 2.2. Corollary 4.3 of Jungck [6] is a special case of Corollary 2.2.

Theorem 2.2. Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. Assume that there exist $S, T \in A_{gf}$ satisfying
\[d(Sx, Ty) < \delta(\cup_{h \in H_{gf}} \{hx, hy\})\]
for $Sx \neq Ty$. Then $H_{gf}$ has a unique fixed point.

Proof. It follows from (ii) of Lemma 1.1 that $gf \in H_{gf}$. The remaining portion of the proof can be derived as in Theorem 2.1. \[\square\]

Theorem 2.3. Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ such that $gf$ is continuous. Assume that for every compact subset $Y$ of $X$ which contains more than one element and is mapped into itself by $gf$, there exist $S, T \in A_{gf}$ satisfying
\[(2.2) \quad d(Sx, Ty) < \delta(Y)\]
for all $x, y$ in $Y$. Then $H_{gf}$ has a unique fixed point.

Proof. Let $A = \cap_{n \in N} (gf)^n X$. By (i) and (ii) of Lemma 1.1, $A$ is a nonempty compact subset of $X$ and $gf \in A_{gf}$. Suppose that $\delta(A) > 0$. Then there exist $a, b \in A$ such that $\delta(A) = d(a, b)$. Since $SA = A = TA$, we can find $x, y \in A$ such that $Sx = a$ and $Ty = b$. By (2.2), we have
\[0 < \delta(A) = d(Sx, Ty) < \delta(A),\]
which is a contradiction. Hence $\delta(A) = 0$, i.e., $A$ is a singleton. The remaining portion of the proof can be derived as in Theorem 2.1. This completes the proof. \[\square\]

Theorem 2.4. Let $f$ and $g$ be continuous self maps of a compact metric space $(X, d)$. Assume that there exist $S \in A_f$ and $T \in A_g$ satisfying
\[(2.3) \quad d(Sx, Ty) < \delta(\cup_{u \in H_f} u O(x, f), \cup_{v \in H_g} v O(y, g))\]
for $Sx \neq Ty$. Then $f$, $g$, $S$ and $T$ have a unique common fixed point which is a unique common fixed point of $H_f$ and $H_g$. 
Proof. Let $A = \cap_{n \in \mathbb{N}} f^n X$ and $B = \cap_{n \in \mathbb{N}} g^n X$. By (i) and (ii) of Lemma 1.1, $A$ and $B$ are nonempty compact subsets of $X$ and $f A = A$, $g B = B$. Thus there exists $a \in A$ and $b \in B$ such that $\delta(A, B) = d(a, b)$. Note that $SA = A$ and $TB = B$. Then there exist $x \in A$ and $y \in B$ such that $S x = a$ and $T y = b$. Suppose that $a \neq b$. Then by (2.3),

$$d(a, b) = d(S x, T y)$$

$$< \delta(\cup_{u \in H_f} u O(x, f), \cup_{v \in H_g} v O(y, g))$$

$$\leq \delta(\cup_{u \in H_f} u A, \cup_{v \in H_g} v B)$$

$$\leq \delta(A, B) = d(a, b),$$

which is a contradiction. Therefore $a = b$ and $\delta(A, B) = 0$. This implies $A = B = \{w\}$, say. Clearly $w$ is a common fixed point of $H_f$ and $H_g$. Since every common fixed point of $f$ and $S$ belongs to $A = \{w\}$ and $f w = S w = w$, so $f$ and $S$ have a unique common fixed point $w$. Similarly $w$ is also a unique common fixed point of $g$ and $T$. Thus $w$ is a unique common fixed point of $H_f$ and $H_g$. This completes the proof. □

3. Nonunique fixed points

Theorem 3.1. Let $f$ and $g$ be continuous self maps of a compact metric space $(X, d)$ satisfying $f \in A_{gf}$. If $f x \neq g y$ implies

$$d(f x, g y) > \inf\{d(u x, f u x), d(u y, f u y), d(u x, g u x),$$

$$d(u y, g u y), d(h x, h y) \mid u \in H_{gf}, h \in C_f \cap C_g\},$$

then at least one of $f$ or $g$ has a fixed point.

Proof. Let $A = \cap_{n \in \mathbb{N}} (gf)^n X$. By (i) of Lemma 1.1, $A$ is a nonempty compact subset of $X$. It follows from Lemma 1.2 that $g \in A_{gf}$. By the continuity of $f$ and $g$ and compactness of $A$, there exist $a, b \in A$ such that

$$d(a, f a) \leq d(x, f x) \quad \text{and} \quad d(b, g b) \leq d(x, g x)$$

for all $x \in A$. We assume without loss of generality that

$$d(a, f a) \leq d(b, g b)$$
Note that $gA = A$. Then there exists a point $w \in A$ such that $gw = a$. Suppose that $a \neq fa$, i.e., $fa \neq gw$. By (3.1), (3.2) and (3.3) we have

$$d(fa, gw) > \inf \{d(ua, fua), d(uw, fuw), d(ua, gua),$$

$$d(uw, guw), d(ha, hw) \mid u \in H_{gf}, h \in C_f \cap C_g \}$$

$$\geq \inf \{d(a, fa), d(b, gb), d(hgw, hw) \mid u \in H_{gf}, h \in C_f \cap C_g \}$$

$$= \inf \{d(a, fa), d(ghw, hw) \mid u \in H_{gf}, h \in C_f \cap C_g \}$$

$$= d(a, fa),$$

which implies that

$$d(a, fa) = d(fa, gw) > d(a, fa),$$

which is impossible. Hence $a = fa$. This completes the proof. \qed

**Remark 3.1.** The following example demonstrates that Theorem 3.1 is more general than Theorem 4.4 of Jungck [6].

**Example 3.1.** Let $X = \{1, 2, 5\}$ and $d(x, y) = |x - y|$. Define $f, g : X \to X$ by

$$f1 = f2 = g1 = 1 \quad \text{and} \quad f5 = g2 = g5 = 2.$$

Then $f$ and $g$ are self maps of a compact metric space $(X, d)$ such that $gf$ is continuous and $\bigcap_{n \in \mathbb{N}} (gf)^n X = \{1\} = f \cap_{n \in \mathbb{N}} (gf)^n X$. It is now a simple matter to show that

$$0 = \inf \{d(ux, fux), d(uy, fuy), d(ux, gux),$$

$$d(uy, guy), d(hx, hy) \mid u \in H_{gf}, h \in C_f \cap C_g \}$$

$$< d(fx, gy) = 1$$

$$< \delta(\bigcup_{h \in H_{gf}} h O(x, y, f, g)) = 4$$

for $fx \neq gy$. Thus the conditions of the above Corollary 2.1 and Theorem 3.1 are satisfied but Theorems 4.2 and 4.4 of Jungck [6] are not applicable since $fg5 = 1 \neq 2 = gf5$.

**Remark 3.2.** Example 4.4 of Jungck [6] shows that not both $f$ and $g$ of the above Theorem 3.1 need have a fixed point and that the fixed point may not be unique.

The proof of the following result goes in a similar fashion as that of Theorem 3.1, so we omit the proof.
Theorem 3.2. Let $f$ and $g$ be self maps of a compact metric space $(X, d)$ satisfying $gf$ is continuous. Assume that there exist $S, T \in A_{gf}$ such that $S$ and $T$ are continuous and
\[
\begin{align*}
    d(Sx, Ty) &> \inf \{ d(ux, Sux), d(uy, Suy), d(ux, Tux), \\
    &\quad d(u, Tuy), d(x, y) \mid u \in H_{gf} \}
\end{align*}
\]
for $Sx \neq Ty$. Then at least one of $S$ or $T$ has a fixed point.

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