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INEQUALITIES OF HLAWKA'S TYPE IN n-INNER PRODUCT SPACES

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Abstract. In this paper, we give Hlawka's type inequalities and related results in n-inner product spaces.

1. Introduction

Let n be a natural number greater than 1. Let X be a linear space of dimension greater or equal to n and $(\cdot,\cdot|\cdot,\cdots,\cdot)$ be a real-valued function on $X^{n+1} = \underbrace{X \times X \times \cdots \times X}_{n+1 \text{ times}}$ satisfying the following conditions:

(nI$_1$) (i) $(a, a|a_2, \cdots, a_n) \geq 0$,
(ii) $(a, a|a_2, \cdots, a_n) = 0$ if and only if $a, a_2, \cdots, a_n$ are linearly dependent,

(nI$_2$) $(a, b|a_2, \cdots, a_n) = (b, a|a_2, \cdots, a_n)$,

(nI$_3$) $(a, b|a_{i_2}, \cdots, a_{i_n}) = (a, b|a_2, \cdots, a_n)$ for every permutation $(i_2, \cdots, i_n)$ of $(2, \cdots, n)$,

(nI$_4$) $(a, a|a_2, a_3, \cdots, a_n) = (a_2, a_2|a, a_3, \cdots, a_n)$,

(nI$_5$) $(\alpha a, b|a_2, \cdots, a_n) = \alpha(a, b|a_2, \cdots, a_n)$ for every real number α,

(nI$_6$) $(a + a', b|a_2, \cdots, a_n) = (a, b|a_2, \cdots, a_n) + (a', b|a_2, \cdots, a_n)$.

Then $(\cdot,\cdot|\cdot,\cdots,\cdot)$ is called an n-inner product on X and $(X, (\cdot,\cdot|\cdot,\cdots,\cdot))$ is called an n-inner product space.

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For example, if X is an inner product space with inner product $(\cdot|\cdot)$, then the function $(\cdot,\cdot|\cdot,\cdots,\cdot)$ defined on X^{n+1} by

$$(a,b|a_2,\cdots,a_n)=\begin{vmatrix} (a\mid b) & (a\mid a_2) & \ldots & (a\mid a_n)\\ (a_2\mid b) & (a_2\mid a_2) & \ldots & (a_2\mid a_n)\\ \vdots & \vdots & \ddots & \vdots\\ (a_n\mid b) & (a_n\mid a_2) & \cdots & (a_n\mid a_n)\end{vmatrix}$$

is an n-inner product on X. Under the same assumptions on X, let $\|\cdot,\cdots,\cdot\|$ be a real-valued function defined on X^n and satisfying the conditions:

- (nN_1) $\|a_1,\cdots,a_n\|=0$ if and only if $a_1,\cdots,a_n$ are linearly dependent,
- (nN_2) $\|a_1,\cdots,a_n\|=\|a_{i_1},\cdots,a_{i_n}\|$ for every permutation $(i_1,\cdots,i_n)$ of $(1,\cdots,n)$,
- (nN_3) $\|\alpha a_1,\cdots,a_n\|=|\alpha|\|a_1,\cdots,a_n\|$ for every real number α,
- (nN_4) $\|a_1+a_1',a_2,\cdots,a_n\|\le\|a_1,a_2,\cdots,a_n\|+\|a_1',a_2,\cdots,a_n\|$.

Then $\|\cdot,\cdots,\cdot\|$ is called an n-norm on X and $(X,\|\cdot,\cdots,\cdot\|)$ is called a linear n-normed space.

If an n-inner product space $(X,(\cdot,\cdot|\cdot,\cdots,\cdot))$ is given, then, for any $a,b,a_2,\cdots,a_n\in X$, we have the following extension of Cauchy-Buniakowski's inequality

$$|(a,b|a_2,\cdots,a_n)|\le\sqrt{(a,a|a_2,\cdots,a_n)}\sqrt{(b,b|a_2,\cdots,a_n)}. \tag{1.1}$$

Moreover, using (nI_1)∼(nI_6) and (1.1), it is easy to see that the formula

$$\|a_1,a_2,\cdots,a_n\|=\sqrt{(a_1,a_1\mid a_2,\cdots,a_n)} \tag{1.2}$$

defines an n-norm on X. For this n-norm, we have

$$(a,b|a_2\cdots,a_n)=\frac{1}{4}\left[\|a+b,a_2\cdots,a_n\|^2-\|a-b,a_2,\cdots,a_n\|^2\right]$$

and the following extension of the parallelogram law is also valid

$$\begin{aligned}&\|a+b,a_2,\cdots,a_n\|^2+\|a-b,a_2,\cdots,a_n\|^2\\&\quad=2\left[\|a,a_2,\cdots,a_n\|^2+\|b,a_2,\cdots,a_n\|^2\right].\end{aligned} \tag{1.3}$$

The details on the definitions and results stated above as well as some further results holding in n-inner product spaces can be found in the book [1, Chapter 12].

The well-known Hlawka's inequality (See [2, p. 521]) is the following inequality

$$\begin{aligned} &\|a\| + \|b\| + \|c\| - \|b+c\| \\ &- \|c+a\| - \|a+b\| + \|a+b+c\| \geq 0, \end{aligned} \tag{1.4}$$

which holds for any three vectors a, b, c in an m-dimensional Euclidean vector space.

In this paper, we give a version of Hlawka's inequality (1.4) and some results related to this inequality in n-inner product spaces.

2. The main results

Consider an n-inner product space $(X, (\cdot, \cdot|\cdot, \cdots, \cdot))$ and assume that an n-norm on X is defined by the formula (1.2), First we show that the following extension of the parallelogram law (1.3) is valid:

PROPOSITION 1. *For $a, b, c, a_2, \cdots, a_n \in X$, we have*

$$\begin{aligned} &\|a, a_2, \cdots, a_n\|^2 + \|b, a_2, \cdots, a_n\|^2 \\ &\quad + \|c, a_2, \cdots, a_n\|^2 + \|a+b+c, a_2, \cdots, a_n\|^2 \\ &= \|b+c, a_2, \cdots, a_n\|^2 + \|c+a, a_2, \cdots, a_n\|^2 \\ &\quad + \|a+b, a_2, \cdots, a_n\|^2. \end{aligned} \tag{2.1}$$

PROOF. By (1.2), the equality (2.1) is equivalent to the equality

$$\begin{aligned} &(a, a|a_2, \cdots, a_n) + (b, b|a_2, \cdots, a_n) \\ &\quad + (c, c|a_2, \cdots, a_n) + (a+b+c, a+b+c|a_2, \cdots, a_n) \\ &= (b+c, b+c|a_2, \cdots, a_n) + (c+a, c+a|a_2, \cdots, a_n) \\ &\quad + (a+b, a+b|a_2, \cdots, a_n), \end{aligned} \tag{2.2}$$

which is easily proved using (nI_2), (nI_5) and (nI_6). This completes the proof. □

Remark 1. (1) Setting $c = -b$ in (2.2) and applying (1.2), (nI_2) and (nI_5), we get the parallelogram law (1.3).

(2) A consequence of Proposition 1 is the following extension of Klamkin's inequality ([2, p. 523]): For any $a, b, c, a_2, \cdots, a_n \in X$, we have

$$\begin{aligned} &\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| \\ &\quad + \|c, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\| \\ &\leq 2\left[\|a+b, a_2, \cdots, a_n\|^2 + \|b+c, a_2, \cdots, a_n\|^2\right. \\ &\quad \left. + \|c+a, a_2, \cdots, a_n\|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Indeed, we can use the inequality between the arithmetic mean and the quadratic mean

$$\frac{x_1 + x_2 + \cdots + x_k}{k} \leq \left(\frac{x_1^2 + x_2^2 + \cdots + x_k^2}{k}\right)^{\frac{1}{2}}$$

and (2.1) to obtain

$$\begin{aligned} &\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| \\ &\quad + \|c, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\| \\ &\leq 2\left[\|a, a_2, \cdots, a_n\|^2 + \|b, a_2, \cdots, a_n\|^2\right. \\ &\quad \left. + \|c, a_2, \cdots, a_n\|^2 + \|a+b+c, a_2, \cdots, a_n\|^2\right]^{\frac{1}{2}} \\ &= 2\left[\|a+b, a_2, \cdots, a_n\|^2 + \|b+c, a_2, \cdots, a_n\|^2\right. \\ &\quad \left. + \|c+a, a_2, \cdots, a_n\|^2\right]^{\frac{1}{2}}. \end{aligned}$$

Theorem 2. (Hlawka's inequality) *If $a, b, c, a_2, \cdots, a_n \in X$ are given vectors from an n-inner product space X, then*

$$\begin{aligned} &\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| \\ &\quad + \|c, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\| \\ &\geq \|b+c, a_2, \cdots, a_n\| + \|c+a, a_2, \cdots, a_n\| \\ &\quad + \|a+b, a_2, \cdots, a_n\|. \end{aligned} \tag{2.4}$$

Proof. The inequality (2.4) is a simple consequence of (nN_4) and the following extension of Hlawka's identity (See [3]), which is equivalent to (2.1):

$$
\begin{aligned}
&(\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| + \|c, a_2, \cdots, a_n\| \\
&\quad - \|b+c, a_2, \cdots, a_n\| - \|c+a, a_2, \cdots, a_n\| - \|a+b, a_2, \cdots, a_n\| \\
&\quad + \|a+b+c, a_2, \cdots, a_n\|)\,(\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| \\
&\quad + \|c, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\|) \\
&= (\|b, a_2, \cdots, a_n\| + \|c, a_2, \cdots, a_n\| - \|b+c, a_2, \cdots, a_n\|) \\
&\quad \times (\|a, a_2, \cdots, a_n\| - \|b+c, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\|) \\
&\quad + (\|c, a_2, \cdots, a_n\| + \|a, a_2, \cdots, a_n\| - \|c+a, a_2, \cdots, a_n\|) \\
&\quad \times (\|b, a_2, \cdots, a_n\| - \|c+a, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\|) \\
&\quad + (\|a, a_2, \cdots, a_n\| + \|b, a_2, \cdots, a_n\| - \|a+b, a_2, \cdots, a_n\|) \\
&\quad \times (\|c, a_2, \cdots, a_n\| - \|a+b, a_2, \cdots, a_n\| + \|a+b+c, a_2, \cdots, a_n\|)\,.
\end{aligned}
$$

This completes the proof. □

Theorem 3. (Hornich's inequality) *Let* $a, a_2, \cdots, a_n, b_1, \cdots, b_m \in X$ *be given vectors from an* n*-inner product space* X. *If*

$$\sum_{k=1}^{m} b_k = -ta \qquad (t \geq 1), \tag{2.5}$$

then we have

$$
\begin{aligned}
&\sum_{k=1}^{m} (\|b_k + a, a_2 \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) \\
&\qquad \leq (m-2)\|a, a_2 \cdots, a_n\|.
\end{aligned} \tag{2.6}
$$

If $t < 1$ *in* (2.5), *then* (2.6) *need not necessarily hold.*

Proof. From (2.4), it follows that

$$
\begin{aligned}
&\|x + a, a_2, \cdots, a_n\| - \|x, a_2, \cdots, a_n\| \\
&\quad + \|y + a, a_2, \cdots, a_n\| - \|y, a_2, \cdots, a_n\| \\
&\leq \|a, a_2, \cdots, a_n\| + \|x + y + a, a_2, \cdots, a_n\| \\
&\quad - \|x + y, a_2, \cdots, a_n\|
\end{aligned} \tag{2.7}
$$

holds for any two vectors $x, y \in X$. Applying (2.7) with $x = b_1$ and $y = b_2$, we get

$$\begin{aligned}\sum_{k=1}^{m} &(\|b_k + a, a_2, \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) \\ &\leq \|a, a_2, \cdots, a_n\| + \|b_1 + b_2 + a, a_2, \cdots, a_n\| \\ &\quad - \|b_1 + b_2, a_2, \cdots, a_n\| \\ &\quad + \sum_{k=3}^{m} (\|b_k + a, a_2, \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) .\end{aligned}$$

We see that the sum on the left-hand side of (2.6) will not decrease if we replace the vectors $b_1, b_2, b_3, \cdots, b_m$ by the vectors $0, b_1 + b_2, b_3, \cdots, b_m$, respectively. Similarly the new sum will not decrease if we replace the vectors $0, b_1+b_2, b_3, b_4, \cdots, b_m$ by the vectors $0, 0, b_1+b_2+b_3, b_4, \cdots, b_m$, respectively. Proceeding with this procedure, we get

$$\begin{aligned}\sum_{k=1}^{m} &(\|b_k + a, a_2, \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) \\ &\leq \sum_{k=1}^{m} (\|c_k + a, a_2, \cdots, a_n\| - \|c_k, a_2, \cdots, a_n\|) ,\end{aligned}$$

where $c_1 = c_2 = \cdots = c_{m-1} = 0$, $c_m = b_1 + b_2 + \cdots + b_m$. This is equivalent to

$$\begin{aligned}\sum_{k=1}^{m} &(\|b_k + a, a_2, \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) \\ &\leq (m-1)\|a, a_2, \cdots, a_n\| + \|b_1 + \cdots + b_m + a, a_2, \cdots, a_n\| \\ &\quad - \|b_1 + \cdots + b_m, a_2, \cdots, a_n\|.\end{aligned}$$

Finally, applying (2.5), we get

$$\begin{aligned}\sum_{k=1}^{m} &(\|b_k + a, a_2, \cdots, a_n\| - \|b_k, a_2, \cdots, a_n\|) \\ &\leq (m-1)\|a, a_2, \cdots, a_n\| + |1-t|\|a, a_2, \cdots, a_n\| \\ &\quad - |t|\|a, a_2, \cdots, a_n\| \\ &= (m-2)\|a, a_2, \cdots, a_n\|,\end{aligned}$$

since $|1 - t| - |t| = -1$ for $t \geq 1$. This completes the proof. □

Remark 2. The result stated in Theorem 3 is a n-inner product space version of an generalization of Hornich's result from [3] (See [2, pp. 521-522]).

To prove the following generalization of Hlawka's inequality, we need one result from [4] (See also [2, p. 528]):

Lemma 4. *Let D be a commutative and additive semigroup and E be a nonempty subset of D satisfying the condition*:

$$b_i \in E \ (i = 1, 2, \cdots, m), \quad \sum_{i=1}^{m} a_i \in E$$
$$\Longrightarrow \sum_{v=1}^{k} a_{i_v} \in E \ (1 \le i_1 < \cdots < i_k \le m).$$

Further, let G be a commutative and additive group which is totally ordered, which means that G is provided by a totally ordering relation $\le$ such that

$$a, b, c \in G, \quad a < b \quad \Longrightarrow \quad a + c < b + c.$$

For a given function $f : E \to G$, consider the condition:

$$(\mathrm{C}_{m,k}) \qquad \sum_{1 \le i_1 < \cdots < i_k \le m} f(b_{i_1} + \cdots + b_{i_k}) \le \binom{m-2}{k-1} \sum_{i=1}^{m} f(b_i) + \binom{m-2}{k-2} f\left(\sum_{i=1}^{m} b_i\right),$$

where $b_i \in E$ $(1 \le i \le m)$, $\sum_{i=1}^{m} b_i \in E$, $2 \le k < m$ and $m \ge 3$. Then, for any $m = 3, 4, \cdots$ and $k = 2, \cdots, m-1$, we have

(1) *The implication*

$$(\mathrm{C}_{3,2}) \Longrightarrow (\mathrm{C}_{m,k})$$

is valid.

(2) *If D contains the neutral element 0, $0 \in E$ and $f(0) = 0$, then the implication*

$$(\mathrm{C}_{m,k}) \Longrightarrow (\mathrm{C}_{3,2})$$

is also valid.

THEOREM 5. *Let $b_1, \cdots, b_m, a_2, \cdots, a_n \in X$ be given vectors in an n-inner product space X. Then we have*

$$\begin{aligned}\sum_{1 \le i_1 < \cdots < i_k \le m} & \|b_{i_1} + \cdots + b_{i_k}, a_2, \cdots, a_n\| \\ \le & \binom{m-2}{k-2} \left(\frac{m-k}{k-1} \sum_{i=1}^{m} \|b_i, a_2, \cdots, a_n\| \right. \\ & \left. + \left\| \sum_{i=1}^{m} b_i, a_2, \cdots, a_n \right\| \right) \end{aligned} \tag{2.8}$$

for $m = 3, 4, \cdots$ and $k = 2, \cdots, m-1$.

PROOF. Take $a_2, \cdots, a_n$ as parameters and consider the function $f : X \to \mathbb{R}^+$ defined by

$$f(b) = \|b, a_2, \cdots, a_n\|.$$

Then the inequality (2.3) is equivalent to the condition $(C_{3,2})$ of Lemma 4, while the inequality (2.8) is the condition $(C_{m,k})$. By Lemma 4 it is clear that $(C_{3,2}) \Longleftrightarrow (C_{m,k})$. This completes the proof. □

Moreover, using Lemma 4, we get the following generalization of the parallelogram law:

THEOREM 6. *Let $b_1, \cdots, b_m, a_2, \cdots, a_n \in X$ be given vectors in an n-inner product space X. Then we have*

$$\begin{aligned}\sum_{1 \le i_1 < \cdots < i_k \le m} & \|b_{i_1} + \cdots + b_{i_k}, a_2, \cdots, a_n\|^2 \\ \le & \binom{m-2}{k-2} \left(\frac{m-k}{k-1} \sum_{i=1}^{m} \|b_i, a_2, \cdots, a_n\|^2 \right. \\ & \left. + \left\| \sum_{i=1}^{m} b_i, a_2, \cdots, a_n \right\|^2 \right) \end{aligned} \tag{2.9}$$

for $m = 3, 4, \cdots$ and $k = 2, \cdots, m-1$.

PROOF. The equality (2.9) is a simple consequence of (2.1) which can be splitted up in two inequalities with opposite directions and, after that, we only need to apply Lemma 4 successively to the functions defined by

$$f(b) = \|b, a_2, \cdots, a_n\|^2, \quad f(b) = -\|b, a_2, \cdots, a_n\|^2,$$

respectively, where $a_2, \cdots, a_n$ are taken as parameters. This completes the proof. □

Remark 3. For $k = 2$, we get, from (2.8) and (2.9),

$$(m-2)\sum_{k=1}^{m}\|b_k, a_2, \cdots, a_n\| + \left\|\sum_{k=1}^{m} b_k, a_2, \cdots, a_n\right\| \geq \sum_{1\leq i<j\leq m}\|b_i + b_j, a_2, \cdots, a_n\|,$$

$$(m-2)\sum_{k=1}^{m}\|b_k, a_2, \cdots, a_n\|^2 + \left\|\sum_{k=1}^{m} b_k, a_2, \cdots, a_n\right\|^2 = \sum_{1\leq i<j\leq m}\|b_i + b_j, a_2, \cdots, a_n\|^2,$$

respectively.

Remark 4. Note that (2.8) is a generalization of the result obtained by D. Ž. Djoković [5] and D. H. Smiley and M. F. Smiley [6], while (2.7) $\sim$ (2.9) are generalizations of the results obtained by D. D. Adamoviĉ [7], [8] (See also [2]).

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