

## ON $q$ -ANALOGUE OF THE TWISTED $L$ -FUNCTIONS AND $q$ -TWISTED BERNOULLI NUMBERS

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ABSTRACT. The aim of this work is to construct twisted  $q$ - $L$ -series which interpolate twisted  $q$ -generalized Bernoulli numbers. By using generating function of  $q$ -Bernoulli numbers, twisted  $q$ -Bernoulli numbers and polynomials are defined. Some properties of this polynomials and numbers are described. The numbers  $L_q(1-n, \chi, \xi)$  is also given explicitly.

### 1. Introduction

In this section, we aim at giving an elementary introduction to some functions which were found useful in number theory. The most famous are Dirichlet  $L$ -functions. We therefore give Dirichlet  $L$ -functions and  $q$ -analogues of the Dirichlet series. We use the notation of Iwasawa [2], Koblitz [8] and Tsumura [12]. Let  $\chi$  be a Dirichlet character of conductor  $f$ . The  $L$ -series attached to  $\chi$  is defined as follows:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where  $\operatorname{Re} s > 1$ . For  $\chi = 1$ , this is the usual Riemann zeta function. It is well known that  $L(s, \chi)$  may be continued analytically to the whole complex plane, except for a simple pole at  $s = 1$  when  $\chi = 1$ . Hurwitz zeta function is defined as follows:

$$\zeta(s, b) = \sum_{n=0}^{\infty} (b+n)^{-s},$$

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where  $\text{Re } s > 1$  and  $0 < b \leq 1$ . For  $b = 1$ , this is the usual Riemann zeta function. It is well known that  $\zeta(s, b)$  may be continued analytically to the whole complex plane, except for a simple pole at  $s = 1$ .

Iwasawa [2] gave fundamental properties of the generalized Bernoulli numbers and Dirichlet  $L$ -functions in more detail. The definition of ordinary Bernoulli numbers is well known: let  $t$  be an indeterminate and let

$$(1.1) \quad F(t) = \frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The coefficients  $B_n, n \geq 0$ , are called Bernoulli numbers. Let  $x$  be another indeterminate and let

$$(1.2) \quad F(t, x) = F(t)e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

The coefficients  $B_n(x), n \geq 0$ , are called Bernoulli polynomials. The generalized Bernoulli numbers  $B_{n,\chi}$  are defined by

$$(1.3) \quad F_{\chi}(t, x) = \sum_{a=0}^{f-1} \frac{\chi(a)te^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

If  $\chi = \chi^0$ , the principal character ( $f = 1$ ), then (1.3) reduces to (1.1).

Note that when  $\chi = \chi^0$ , the principal character ( $f = 1$ ), we have

$$\sum_{n \in \mathbb{N}} B_{n,1} \frac{t^n}{n!} = \frac{te^t}{e^t - 1} = \frac{t}{e^t - 1} + t,$$

so  $B_{n,1} = B_n$  except for  $n = 1$ , when we have  $B_{1,1} = \frac{1}{2}, B_1 = -\frac{1}{2}$ .

If  $\chi \neq 1$  then  $B_{0,\chi} = 0$ , since  $\sum_{a=1}^d \chi(a) = 0$ . A relationship between  $L(1 - n, \chi)$  and  $B_{n,\chi}$  is given as follows [2]: for  $n$  be a positive integer

$$L(1 - n, \chi) = -\frac{B_{n,\chi}}{n}.$$

In [11], the author constructed an elementary introduction to twisted  $L$ -functions which are found useful in number theory and  $p$ -adic analysis. The author reviewed some of the basic facts about twisted  $L$ -series which interpolated twisted Bernoulli numbers. Their values at negative integers were given in terms of twisted Bernoulli numbers,  $B_{n,\chi,\xi}$ . Finally, the author discussed the value at 1 and analytic continuation of this function. Let  $r$  be a positive integer, and let  $\varepsilon \neq 1$  be any nontrivial  $r$ -th root of 1. Let  $\xi^f = \varepsilon$ . Then twisted  $L$ -functions are defined as

follows:

$$L(s, f, \xi) = \sum_{n=1}^{\infty} \frac{\chi(n)\xi^n}{n^s}.$$

Since the function  $n \rightarrow \chi(n)\xi^n$  has period  $fr$ , this is a special case of the Dirichlet  $L$ -functions considered above. Such  $L$ -series (for  $r = f$ ) are used classically to prove the formula for  $L(1, \chi)$  by Fourier inversion. Koblitz ([7], [8]) gave a relation between  $L(1-n, f, \xi)$  and  $B_{n,\chi,\xi}$ . He also defined  $p$ -adic twisted  $L$ -functions,  $L_p(s, \chi, \xi)$ , where  $s$  is  $p$ -adic number. Using these functions, he constructed  $p$ -adic measures and integration.

In [11], the author defined generalized of the functions  $F(t)$  and  $F(t, x)$ , which are mentioned in the above. Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1 and let  $t$  be an indeterminate.  $F_{\chi,\xi}(t)$  is defined as follows:

$$(1.4) \quad F_{\chi,\xi}(t) = \sum_{a=0}^{f-1} \frac{\chi(a)\xi^a t e^{at}}{\xi^f e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi} \frac{t^n}{n!}.$$

If  $r = 1$ , then (1.4) reduces to (1.3). The coefficients  $B_{n,\chi,\xi}$ ,  $n \geq 0$ , are called twisted Bernoulli numbers. Let  $x$  be another indeterminate and  $F_{\chi,\xi}(t, x)$  is defined as follows:

$$(1.5) \quad F_{\chi,\xi}(t, x) = F_{\chi,\xi}(t) e^{xt} = \sum_{a=0}^{f-1} \frac{\chi(a)\xi^a t e^{(a+x)t}}{\xi^f e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi,\xi}(x) \frac{t^n}{n!}.$$

The coefficients  $B_{n,\chi,\xi}(x)$ ,  $n \geq 0$ , are called twisted Bernoulli polynomials.

**THEOREM 1.** ([11]) *Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1. Then we have*

$$(1.6) \quad F_{\chi,\xi}(t, x) = \frac{1}{rf} \sum_{a=0}^{f-1} \chi(a)\xi^a \sum_{b=0}^{r-1} \xi^{bf} F(trf, \frac{a + bf + x - rf}{rf}).$$

**DEFINITION 1.** ([11]) *Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1. Then we have*

$$(1.7) \quad B_{n,\chi,\xi}(x) = (rf)^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a)\xi^{a+bf} B_n(\frac{a + bf + x - rf}{rf}),$$

and

$$(1.8) \quad B_{n,\chi,\xi} = (rf)^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a)\xi^{a+bf} B_n(\frac{a + bf - rf}{rf}).$$

where  $B_{n,\chi,\xi}(x)$  and  $B_{n,\chi,\xi}$  are twisted generalized Bernoulli polynomials and numbers, respectively.

In [1], Carlitz defined  $q$ -extensions of Bernoulli numbers and polynomials and proved properties generalizing those satisfied by  $B_k$  and  $B_k(x)$ . He defined a set of numbers  $\eta_k = \eta_k(q)$  inductively by

**$q$ -Bernoulli numbers:**  $\eta_0 = 1, (q\eta + 1)^k - \eta_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases}$  with

the usual convention about replacing  $\eta^k$  by  $\eta_k$ . These numbers are  $q$ -analogues of the ordinary Bernoulli numbers  $B_k$ , but they do not remain finite when  $q = 1$ . So he modified the definition as

$$\beta_0 = 1, (q\beta + 1)^k - \beta_k = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

These numbers  $\beta_k = \beta_k(q)$  were called the  $q$ -Bernoulli numbers, which reduce to  $B_k$  when  $q = 1$ . Some properties of  $\beta_k(q)$  were investigated by a lot of authors. Koblitz [9] constructed a  $q$ -analogue of  $p$ -adic Dirichlet  $L$ -series which interpolated Carlitz's  $q$ -Bernoulli numbers at non-positive integers and he raised two questions. In [9], Koblitz gave properties of  $q$ -extension of Bernoulli numbers and polynomials and he constructed  $p$ -adic measure and Dirichlet  $L$  function. In [10], Satoh constructed a complex analytic  $q$ - $L$ -series which is a  $q$ -analogue of Dirichlet's  $L$ -functions and interpolates  $q$ -Bernoulli numbers, which is an answer to Koblitz's question 1. He induced this  $q$ - $L$ -series from the generating function of  $q$ -Bernoulli numbers. Tsumura [12] defined  $q$ - $L$ -series which is slightly different from the one in [10]. He also gave  $q$ -analogues of the Dirichlet  $L$ -series and Dedekind  $\zeta$ -function. In [3], Kim showed that Carlitz's  $q$ -Bernoulli number can be represented as an integral by the  $q$ -analogue  $\mu_q$  of ordinary  $p$ -adic invariant measure and he gave an answer to a part of a question of Koblitz. In [4], Kim gave a proof of the distribution relation for  $q$ -Bernoulli polynomials by using  $q$ -integral and evaluated the values of  $p$ -adic  $q$ - $L$ -function.

Tsumura [12] modified the definition of the  $q$ -Bernoulli numbers  $B_k = B_k(q)$  as follows:  $B_0(q) = \frac{q-1}{\log q}, (qB + 1)^k - B_k(q) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$

The usual convention about replacing  $B_k$  by  $B^k$ . We can see that  $B_k(q) \rightarrow B_k$  when  $q \rightarrow 1$ .

EXAMPLE 1. We give some Tsumura's  $q$ -Bernoulli numbers:

$$B_1(q) = \frac{\log q + 1 - q}{(q - 1) \log q}, B_2(q) = \frac{[2](1 - q) - 2 \log q}{[2](q - 1)^2 \log q}, \dots$$

We now summarize our present paper in detail as follows.

In Section 2,  $q$ -analogues of Dirichlet series are given due to Tsumura [12]. The relation between  $\zeta_q(s, a)$  and  $L_q(s, \chi)$  are proved.

In Section 3, we construct a twisted  $q$ - $L$ -series which interpolates twisted  $q$ -generalized Bernoulli numbers.

In Section 4, we define the generating functions of twisted  $q$ -Bernoulli polynomials and numbers. We prove the numbers  $L_q(1 - n, \chi, \xi)$  which is related to “twisted”  $q$ -generalized Bernoulli numbers.

### 2. $q$ -analogue of the Dirichlet $L$ -series

Let  $\mathbb{R}$  and  $\mathbb{C}$  be the field of real and complex numbers as usual. Let  $q$  be a real number with  $0 < q < 1$ . We denote  $[x] = [x; q] = \frac{1-q^x}{1-q}$ . Note that  $[x; q] \rightarrow x$  if  $q \rightarrow 1$ .  $q$ -analogues of the Dirichlet sires is defined as follows [12]: for a set of complex numbers  $\{c_n\}$ ,

$$f(s) = \sum_{n=1}^{\infty} \frac{c_n q^{-n}}{(q^{-n}[n])^s},$$

for  $s \in \mathbb{C}$ . Tsumura [12] investigated these series by using a method similar to the method used to treat the ordinary Dirichlet series. For example the  $q$ -Riemann  $\zeta$ -function can be defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{-n}}{(q^{-n}[n])^s}.$$

We can see that the right-hand side of this series converges when  $\text{Re}(s) > 1$ . And  $\zeta_q(s)$  may be analytically continued to the whole complex plane, except for a simple pole at  $s = 1$  with residue  $\frac{q-1}{\log q}$  ( for detail see [12]).

DEFINITION 2. ( $q$ -analogue of the Hurwitz  $\zeta$ -functions [12]).

$$(2.1) \quad \zeta_q(s, b) = q^{s-1} \sum_{n=0}^{\infty} \frac{q^{-n}}{(q^{-n}[n] + b)^s},$$

for  $0 < b \leq 1$ , and  $s \in \mathbb{C}$ .  $\zeta_q(s, b) \rightarrow \zeta(s, b)$  if  $q \rightarrow 1$ , where  $\zeta(s, b)$  is the ordinary Hurwitz  $\zeta$ -function.

PROPOSITION 1. ([12]) If  $k \geq 1$  and  $0 < b \leq 1$ , then

$$(2.2) \quad \zeta_q(1 - k, b) = \frac{(-1)^{k+1} B_k(b, q)}{kq^k}.$$

DEFINITION 3. ( $q$ -analogue of the Dirichlet  $L$ -series [12]). Let  $\chi$  be a Dirichlet character of conductor  $f$ .

$$L_q(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)q^{-n}}{(q^{-n}[n])^s},$$

for  $s \in \mathbb{C}$ .  $L_q(s, \chi) \rightarrow L(s, \chi)$  if  $q \rightarrow 1$ .

For  $\chi$  as above, the generalized  $q$ -Bernoulli numbers are defined as follows [12]:

$$B_{k,\chi}(q) = [f]^{k-1} \sum_{a=1}^f \chi(a)q^{ak} B_k\left(\frac{[a]}{[f]}, q^f\right),$$

for  $k \geq 1$ . In the case when  $\chi = 1$ ,  $B_{k,1}(q) = q^{-k}B_k(1, q) = B_k(q)$ .

Tsumura [12] gave a connection between  $L_q(s, \chi)$  and  $\zeta_q(s, a)$  as follows:

THEOREM 2. Let  $\chi$  be a Dirichlet character of conductor  $f$ .

$$L_q(s, \chi) = [f]^{-s} \sum_{a=1}^f \chi(a)q^{(1-s)(f-a)} \zeta_{q^f}\left(s, \frac{[a]}{[f]}\right),$$

for  $s \in \mathbb{C}$  and  $s > 1$ .

*Proof.* Substituting  $n = mf + a$ , where  $m = 0, 1, 2, \dots, \infty$ , and  $a = 1, 2, \dots, f$  into definition of  $L_q(s, \chi)$  as above, we obtain

$$\begin{aligned} L_q(s, \chi) &= \sum_{a=1}^f \chi(a) \sum_{n=0}^{\infty} \frac{q^{-a-mf}}{(q^{-a-mf}[a + mf])^s} \\ &= \sum_{a=1}^f \chi(a)q^{as-a} \sum_{n=0}^{\infty} \frac{q^{-mf}}{\left(\frac{q^{-mf}-q^a}{1-q}\right)^s} \\ &= \sum_{a=1}^f \chi(a)q^{as-a} \sum_{n=0}^{\infty} \frac{q^{-mf}}{\left\{q^{-mf}\left(\frac{1-q^{mf}}{1-q^f}\right)\left(\frac{1-q^f}{1-q}\right) + \left(\frac{1-q^a}{1-q}\right)\right\}^s} \\ &= \sum_{a=1}^f \chi(a)q^{as-a} \sum_{n=0}^{\infty} \frac{q^{-mf}}{\{q^{-mf}[m; q^f][f] + [a]\}^s} \\ &= [f]^{-s} \sum_{a=1}^f \chi(a)q^{as-a-fs+f} \zeta_{q^f}\left(s, \frac{[a]}{[f]}\right). \end{aligned}$$

where

$$(2.3) \quad q^a \left( \frac{1 - q^{mf}}{1 - q^f} \right) \left( \frac{1 - q^f}{1 - q} \right) + \left( \frac{1 - q^a}{1 - q} \right) = [m; q^f][f]q^a + [a] = [a + mf].$$

Therefore we obtain the desired result. □

We wish to give the numbers  $L_q(1 - n, \chi, \xi)$  explicitly in the later section. For this we need the  $q$ -twisted Bernoulli numbers, which are defined below.

### 3. $q$ -twisted $L$ -functions

Our primary goal in this section is to construct  $q$ -twisted  $L$ -functions which interpolate  $q$ -twisted generalized Bernoulli numbers  $B_{n,\chi,\xi}(q)$ . We discuss some of the fundamental properties of these numbers which are needed in the later section.

By using the definition of  $L_q(s, \chi)$  and  $L(s, \chi, \xi)$ , we can define a  $q$ -analogue of twisted  $L$ -function.

DEFINITION 4. ( $q$ -analogue of the twisted  $L$ -functions). Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1.

$$(3.1) \quad L_q(s, \chi, \xi) = \sum_{n=1}^{\infty} \frac{\chi(n)\xi^n q^{-n}}{(q^{-n}[n])^s},$$

for  $s \in \mathbb{C}$ .  $L_q(s, \chi, \xi) \rightarrow L(s, \chi, \xi)$  if  $q \rightarrow 1$ . Since the function  $n \rightarrow \chi(n)\xi$  has period  $fr$ , this is a special case of the Dirichlet  $L_q$ -functions considered above.

REMARK 1. Koblitz ([7], [8]) defined  $p$ -adic twisted  $L$ -functions,  $L_p(s, \chi, \xi)$ , where  $s$  is  $p$ -adic number. Using these functions, he constructed  $p$ -adic measures and integration, neither of which we have include here. In [9], Koblitz constructed a  $q$ -analogue of the  $p$ -adic  $L$ -function  $L_{p,q}(s, \chi)$  which interpolated Carlitz's  $q$ -Bernoulli numbers.  $q$ -analogue of the  $p$ -adic twisted  $L$ -function  $L_{p,q}(s, \chi, \xi)$  may be defined. We have also omitted a discussion of  $p$ -adic case.

Now, we give a relation between  $L_q(s, \chi, \xi)$  and  $\zeta_q(s, a)$  as follows.

**THEOREM 3.** *Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1.*

$$(3.2) \quad L_q(s, \chi, \xi) = [rf]^{-s} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} q^{(1-s)(2rf-a-bf)} \cdot \zeta_{q^{rf}}\left(s, \frac{[a+bf-rf]}{[rf]}\right),$$

for  $s \in \mathbb{C}$ .

*Proof.* Substituting  $n = a + bf - rf + rfm$  with  $m = 0, 1, \dots, \infty$ ,  $a = 1, 2, \dots, f - 1$ , and  $b = 1, 2, \dots, r - 1$  into (3.1), we obtain

$$L_q(s, \chi, \xi) = \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \sum_{m=0}^{\infty} \chi(a) \xi^{a+bf} \frac{q^{-(a+bf-rf+rfm)}}{(q^{-(a+bf-rf)} q^{-rfm} [a+bf-rf+rfm])^s}.$$

After some calculations we get

$$L_q(s, \chi, \xi) = \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} q^{as-a+bf s-bf+rf-rfs} \cdot \sum_{m=0}^{\infty} \frac{q^{-rfm}}{(q^{-rfm} \frac{1-q^{a+bf-rf+rfm}}{1-q})^s}.$$

By using (2.3) and (2.1) in the above, we obtain

$$L_q(s, \chi, \xi) = [rf]^{-s} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} q^{(1-s)(2rf-bf-a)} \cdot \zeta_{q^{rf}}\left(s, \frac{[a+bf-rf]}{[rf]}\right).$$

We obtain the desired result. □

#### 4. $q$ -twisted Bernoulli numbers and polynomials

The main purpose of this section is to give the numbers  $L_q(1-n, \chi, \xi)$  which is related to twisted  $q$ -generalized Bernoulli numbers. Some basic facts about  $F_q(t)$  and  $L_q$ -series are reviewed. Then their values at negative integers are given in terms of twisted  $q$ -generalized Bernoulli numbers.

We define the following  $F_q(t)$  function which is similar to the one in [12]. The generating function of  $q$ -Bernoulli numbers  $F_q(t)$  is given by

$$(4.1) \quad F_q(t) = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!} = \sum_{n=0}^{\infty} tq^{-n} e^{-q^{-n}[n]t}.$$

The remarkable point is that the series on the right-hand side of (4.1) is uniformly convergent in the wider sense. Hence we have

$$B_k(q) = \frac{d^k}{dt^k} F_q(t).$$

This is used to construct a  $q$ -Dirichlet series which are given above. By using this idea, Satoh [10] constructed the complex  $q$ - $L$ -series which interpolated Carlitz's  $q$ -Bernoulli numbers  $\beta_n(q)$ . Higher order of the  $q$ -Bernoulli numbers and polynomials,  $\beta_n^{(-m,k)}(q)$ , for  $m, k \in \mathbb{N}$ , are defined by Kim [5], Kim and Rim [6]. They gave relations between these numbers and  $L_{q,p}$ -series (see for detail [5], [6]). Tsumura [12] studied a  $q$ -analogue of the Dirichlet  $L$ -series which interpolated Tsumura's  $q$ -Bernoulli numbers  $B_n(q)$ .

We shall explicitly determine the generating function  $F_q(t)$  of  $B_k(q)$  :

$$F_q(t) = \sum_{k=0}^{\infty} B_k(q) \frac{t^k}{k!}.$$

This is the unique solution of the following  $q$ -difference equation:

$$(4.2) \quad F_q(t) = e^t F_q(qt) - qte^t.$$

LEMMA 1.

$$(4.3) \quad F_q(t) = \sum_{k=0}^{\infty} tq^{-n} e^{-q^{-n}[n]t}.$$

*Proof.* The right hand side is uniformly convergent in the wider sense, and satisfies (4.2). □

REMARK 2. i) By using (2.1) and (4.3), then we arrive at proof of (2.2). ii) As  $q \rightarrow 1$  in (4.3), we have  $F_q(t) \rightarrow F(t)$  in (1.1).

THEOREM 4. Let  $k > 0$ ,  $\zeta_q(1 - k) = -\frac{(-1)^k B_k(q)}{k}$ .

*Proof.* By using definition of  $\zeta_q(s)$  and Lemma1, we obtain

$$B_k(q) = \frac{d^k}{dt^k} F_q(t) = (-1)^{k-1} k \zeta_q(1 - k),$$

for  $k > 0$ . So we obtain the desired result. □

The generating function of  $q$ -Bernoulli polynomials  $F_q(t, x)$  is defined by

$$F_q(t, x) = \sum_{k=0}^{\infty} tq^{-k} e^{(-q^{-k}[k]+[x])t}.$$

As  $q \rightarrow 1$  in, we have  $F_q(t) \rightarrow F(t)$  in (1.2).

Let  $\chi$  be a Dirichlet character of conductor  $f$ . Then we define the following  $F_{q,\chi}(t, x)$  function which is generating  $q$ -generalized Bernoulli polynomials  $B_{q,\chi}(q, x)$ .

DEFINITION 5.

$$(4.4) \quad F_{q,\chi}(t, x) = \frac{1}{[f]} \sum_{a=0}^f \chi(a) F_q(t[f], \frac{[a - f + x]}{[f]}).$$

LEMMA 2. Let  $\chi$  be a Dirichlet character of conductor  $f$ .

$$F_{q,\chi}(t, x) = t \sum_{a=0}^{f-1} \chi(a) e^{([a]+[x]q^a - [f]q^{a+x-f})t} \sum_{k=0}^{\infty} q^{-k} e^{-q^{-k}[k][f]t}.$$

*Proof.* By using (4.4) and (2.3) ( $[a + x] = [a] + [x]q^a$  and  $[fa] = [f][a; q^f]$ ), we obtain

$$F_{q,\chi}(t, x) = t \sum_{a=0}^{f-1} \chi(a) \sum_{k=0}^{\infty} q^{-k} e^{(-q^{-k}[k][f]+[a]+[x]q^a - [f]q^{a+x-f})t}.$$

After some elementary calculations, we get the desired result. □

REMARK 3. As  $q \rightarrow 1$ , we have  $F_{q,\chi}(t, x) \rightarrow F_{\chi}(t, x)$  in (1.3).

By using the definition of  $F_{q,\chi}(t, x)$ , we can define a twisted generating function of twisted  $q$ -Bernoulli polynomials.

DEFINITION 6. Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1.

$$(4.5) \quad F_{q,\chi,\xi}(t, x) = \frac{1}{[rf]} \sum_{a=0}^{f-1} \chi(a) \xi^a \sum_{b=0}^{r-1} \xi^{bf} F_q(t[rf], \frac{[a - rf + bf + x]}{[rf]}).$$

REMARK 4. As  $q \rightarrow 1$ , we have  $F_{q,\chi,\xi}(t, x) \rightarrow F_{\chi,\xi}(t, x)$  in (1.5).

By using (1.6), (1.7), (1.8) and (4.5), we can define a twisted  $q$ -Bernoulli numbers  $B_{k,\chi,\xi}(q)$ .

DEFINITION 7. Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1.

$$B_{k,\chi,\xi}(q) = \frac{1}{[rf]^{1-k}} \sum_{a=0}^{f-1} \chi(a) \xi^a q^{-ak} \sum_{b=0}^{r-1} \xi^{bf} q^{-bfk} B_k\left(\frac{[a - rf + bf]}{[rf]}, q^{rf}\right).$$

We shall next describe some properties of  $B_{n,\chi,\xi}(x, q)$  and  $B_{n,\chi,\xi}(q)$  as follows:

i) if  $\chi = \chi^0$ , the principal character ( $f = 1$ ), and  $r = 1$ , then

$$F_{q,\chi,\xi}(t, x) = F_q(t, x)$$

and

$$F_{\chi,\xi}(t, x) = F_q(t, [x]),$$

so that

$$\begin{aligned} B_{n,\chi^0,1}(x, q) &= B_n(x, q), \\ B_{n,\chi^0,1} &= B_n(q), n \geq 0. \end{aligned}$$

ii)

$$B_{n,\chi,\xi}(0, q) = B_{n,\chi,\xi}(q), n \geq 0.$$

iii)

$$\begin{aligned} B_{0,\chi,\xi}(q) &= [rf]^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} B_0 \\ &= \frac{q-1}{\log q} [rf]^{n-1} \sum_{a=0}^{f-1} \chi(a) \xi^a \sum_{b=0}^{r-1} \xi^{bf} \\ &= \frac{q-1}{\log q} [rf]^{n-1} \sum_{a=0}^{f-1} \chi(a) \xi^a \frac{\xi^{rf} - 1}{\xi^f - 1} \\ &= 0. \end{aligned}$$

Thus we have

$$B_{0,\chi,1}(q) = \frac{q-1}{\log q} [f]^{n-1} \sum_{0 \leq a < f} \chi(a) = 0.$$

Hence,

$$\deg(B_{n,\chi,\xi}(x, q)) < n$$

if  $\chi = \chi^0$ , the principal character ( $f = 1$ ) and  $r = 1$ .

iv) If  $r = 1$ , then

$$B_{n,\chi,1}(x, q) = B_{n,\chi}(x, q),$$

and

$$B_{n,\chi,\xi}(q) = B_{n,\chi}(q), n \geq 0.$$

We now give a relation between twisted  $q$ -Bernoulli numbers and  $q$ -twisted  $L$ -functions as follows.

**THEOREM 5.** *Let  $\chi$  be a Dirichlet character with conductor  $f$  and let  $\xi$  be  $r$ th root of 1. Let  $n \geq 1$ . Then*

$$L_q(1-n, \chi, \xi) = (-1)^{n+1} q^{rfk} \frac{B_{n,\chi,\xi}(q)}{n}.$$

*Proof.* Setting  $s = 1 - n$  in (3.2), we have

$$\begin{aligned} L_q(1-n, \chi, \xi) &= [rf]^{n-1} \sum_{a=0}^{f-1} \sum_{b=0}^{r-1} \chi(a) \xi^{a+bf} q^{n(2rf-a-bf)} \\ &\quad \cdot \zeta_{q^{rf}}(1-n, \frac{[a+bf-rf]}{[rf]}). \end{aligned}$$

Writing  $q \rightarrow q^{rf}$  and  $b \rightarrow \frac{[a+bf-rf]}{[rf]}$  in (2.2) and substituting this result into the above equation, we arrive at the desired result.  $\square$

**REMARK 5.**  $L_q(s, \chi, \xi)$  values at  $s = 1$  may be calculate and relations with class numbers may be found. We do not discuss these properties here.

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