WEAK SMOOTH $\alpha$-STRUCTURE OF SMOOTH TOPOLOGICAL SPACES

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Abstract. In [3] and [6] the concepts of smooth closure, smooth interior, smooth $\alpha$-closure and smooth $\alpha$-interior of a fuzzy set were introduced and some of their properties were obtained. In this paper, we introduce the concepts of several types of weak smooth compactness and weak smooth $\alpha$-compactness in terms of these concepts introduced in [3] and [6] and investigate some of their properties.

1. Introduction

Badard [1] introduced the concept of a smooth topological space which is a generalization of Chang’s fuzzy topological space [2]. Many mathematical structures in smooth topological spaces were introduced and studied. In particular, Gayyar, Kerre and Ramadan [5] and Demirci [3, 4] introduced the concepts of smooth closure and smooth interior of a fuzzy set and several types of compactness in smooth topological spaces and obtained some of their properties. In [6] we introduced the concepts of smooth $\alpha$-closure and smooth $\alpha$-interior of a fuzzy set which are generalizations of smooth closure and smooth interior of a fuzzy set defined in [3] and obtained some of their properties. In this paper, we introduce the concepts of several types of weak smooth compactness and weak smooth $\alpha$-compactness in terms of these concepts introduced in [3] and [6] and investigate some of their properties.
2. Preliminaries

Let \( X \) be a set and \( I = [0, 1] \) be the unit interval of the real line. \( I^X \) will denote the set of all fuzzy sets of \( X \). \( 0_X \) and \( 1_X \) will denote the characteristic functions of \( \phi \) and \( X \), respectively.

A smooth topological space (s.t.s.) \([7]\) is an ordered pair \((X, \tau)\), where \( X \) is a non-empty set and \( \tau : I^X \rightarrow I \) is a mapping satisfying the following conditions:

1. \( \tau(0_X) = \tau(1_X) = 1; \)
2. \( \forall A, B \in I^X, \ \tau(A \cap B) \geq \tau(A) \land \tau(B); \)
3. \( \forall A \in I^X, \ \tau(\cup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i). \)

Then the mapping \( \tau : I^X \rightarrow I \) is called a smooth topology on \( X \). The number \( \tau(A) \) is called the degree of openness of \( A \).

A mapping \( \tau^* : I^X \rightarrow I \) is called a smooth cotopology \([7]\) if the following three conditions are satisfied:

1. \( \tau^*(0_X) = \tau^*(1_X) = 1; \)
2. \( \forall A, B \in I^X, \ \tau^*(A \cup B) \geq \tau^*(A) \land \tau^*(B); \)
3. \( \forall A \in I^X, \ \tau^*(\cap_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau^*(A_i). \)

If \( \tau \) is a smooth topology on \( X \), then the mapping \( \tau^* : I^X \rightarrow I \), defined by \( \tau^*(A) = \tau(\overline{A}) \) where \( \overline{A} \) denotes the complement of \( A \), is a smooth cotopology on \( X \). Conversely, if \( \tau^* \) is a smooth cotopology on \( X \), then the mapping \( \tau : I^X \rightarrow I \), defined by \( \tau(A) = \tau^*(A^c) \), is a smooth topology on \( X \) \([7]\).

For the s.t.s. \((X, \tau)\) and \( \alpha \in [0, 1]\), the family \( \tau_\alpha = \{A \in I^X : \tau(A) \geq \alpha\} \) defines a Chang’s fuzzy topology (CFT) on \( X \) \([2]\). The family of all closed fuzzy sets with respect to \( \tau_\alpha \) is denoted by \( \tau_\alpha^* \) and we have \( \tau_\alpha^* = \{A \in I^X : \tau^*(A) \geq \alpha\} \). For \( A \in I^X \) and \( \alpha \in [0, 1]\), the \( \alpha \)-closure (resp., \( \alpha \)-interior) of \( A \), denoted by \( cl_\alpha(A) \) (resp., \( int_\alpha(A) \)), is defined by \( cl_\alpha(A) = \bigcap \{K \in \tau^*_\alpha : A \subseteq K\} \) (resp., \( int_\alpha(A) = \bigcup \{K \in \tau_\alpha : K \subseteq A\}\)).

Demiirci \([3]\) introduced the concepts of smooth closure and smooth interior in smooth topological spaces as follows:

Let \((X, \tau)\) be a s.t.s. and \( A \in I^X \). Then the \( \tau \)-smooth closure (resp., \( \tau \)-smooth interior) of \( A \), denoted by \( \overline{A} \) (resp., \( A^\circ \)), is defined by \( \overline{A} = \bigcap \{K \in I^X : \tau(K) > 0, A \subseteq K\} \) (resp., \( A^\circ = \bigcup \{K \in I^X : \tau(K) > 0, K \subseteq A\}\)).

Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \( f : X \rightarrow Y \) is called smooth continuous with respect to \( \tau \) and \( \sigma \) \([7]\) if \( \tau(f^{-1}(A)) \geq \sigma(A) \) for every \( A \in I^Y \). A function \( f : X \rightarrow Y \) is called weakly smooth continuous with respect to \( \tau \) and \( \sigma \) \([7]\) if \( \sigma(A) > 0 \Rightarrow \tau(f^{-1}(A)) > 0 \) for every \( A \in I^Y \). In this paper, a weakly smooth
continuous function with respect to $\tau$ and $\sigma$ is called a quasi-smooth continuous function with respect to $\tau$ and $\sigma$.

A function $f : X \to Y$ is smooth continuous with respect to $\tau$ and $\sigma$ if and only if $\tau^*(f^{-1}(A)) \geq \sigma^*(A)$ for every $A \in I^Y$. A function $f : X \to Y$ is weakly smooth continuous with respect to $\tau$ and $\sigma$ if and only if $\sigma^*(A) > 0 \Rightarrow \tau^*(f^{-1}(A)) > 0$ for every $A \in I^Y$ [7].

**Theorem 2.1** [3]. Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces. If a function $f : X \to Y$ is quasi-smooth continuous with respect to $\tau$ and $\sigma$, then

(a) $f(A) \subseteq f(A)$ for every $A \in I^X$,
(b) $f^{-1}(A) \subseteq f^{-1}(A)$ for every $A \in I^Y$,
(c) $f^{-1}(A^\sigma) \subseteq (f^{-1}(A))^\sigma$ for every $A \in I^Y$.

A function $f : X \to Y$ is called smooth open (resp., smooth closed) with respect to $\tau$ and $\sigma$ [7] if $\tau(A) \leq \sigma(f(A))$ (resp., $\tau^*(A) \leq \sigma^*(f(A))$) for every $A \in I^X$.

**Theorem 2.2** [3]. Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and $A \in I^X$. If a function $f : X \to Y$ is smooth open with respect to $\tau$ and $\sigma$, then $f(A^\sigma) \subseteq (f(A))^\sigma$.

A function $f : X \to Y$ is called smooth preserving (resp., strict smooth preserving) with respect to $\tau$ and $\sigma$ [5] if $\sigma(A) \geq \sigma(B) \iff \tau(f^{-1}(A)) \geq \tau(f^{-1}(B))$ (resp., $\sigma(A) > \sigma(B) \iff \tau(f^{-1}(A)) > \tau(f^{-1}(B))$) for every $A, B \in I^Y$.

If $f : X \to Y$ is a smooth preserving function (resp., a strict smooth preserving function) with respect to $\tau$ and $\sigma$, then $\sigma^*(A) \geq \sigma^*(B) \iff \tau^*(f^{-1}(A)) \geq \tau^*(f^{-1}(B))$ (resp., $\sigma^*(A) > \sigma^*(B) \iff \tau^*(f^{-1}(A)) > \tau^*(f^{-1}(B))$) for every $A, B \in I^Y$ [5].

A function $f : X \to Y$ is called smooth open preserving (resp., strict smooth open preserving) with respect to $\tau$ and $\sigma$ [5] if $\tau(A) \geq \tau(B) \Rightarrow \sigma(f(A)) \geq \sigma(f(B))$ (resp., $\tau(A) > \tau(B) \Rightarrow \sigma(f(A)) > \sigma(f(B))$) for every $A, B \in I^X$.

### 3. Types of weak smooth compactness

Demirci [4] defined the families $W(\tau) = \{A \in I^X : A = A^o\}$ and $W^*(\tau) = \{A \in I^X : A = \overline{A}\}$, where $(X, \tau)$ is a s.t.s. Note that $A \in W(\tau) \iff A^c \in W^*(\tau)$. In this section, we introduce topological concepts of a s.t.s. in terms of the family $W(\tau)$ and investigate some of their properties.
Definition 3.1 [4]. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \(f : X \to Y\) is called weak smooth continuous with respect to \(\tau\) and \(\sigma\) if \(A \in W(\sigma) \Rightarrow f^{-1}(A) \in W(\tau)\) for every \(A \in \mathcal{I}^Y\).

Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \(f : X \to Y\) is weak smooth continuous with respect to \(\tau\) and \(\sigma\) if and only if \(A \in W^*(\sigma) \Rightarrow f^{-1}(A) \in W^*(\tau)\) for every \(A \in \mathcal{I}^Y\) [4].

Definition 3.2 [4]. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces. A function \(f : X \to Y\) is called weak smooth open (resp., weak smooth closed) with respect to \(\tau\) and \(\sigma\) if \(A \in W(\tau) \Rightarrow f(A) \in W(\sigma)\) (resp., \(A \in W^*(\tau) \Rightarrow f(A) \in W^*(\sigma)\)) for every \(A \in \mathcal{I}^X\).

Definition 3.3 [4]. A s.t.s. \((X, \tau)\) is called weak smooth compact if every family in \(W(\tau)\) covering \(X\) has a finite subcover.

Note that a weak smooth compact s.t.s. \((X, \tau)\) is smooth compact.

Theorem 3.4 [4]. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and \(f : X \to Y\) a surjective and weak smooth continuous function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is weak smooth compact, then so is \((Y, \sigma)\).

Proof. Let \(f : X \to Y\) be a quasi-smooth continuous function with respect to \(\tau\) and \(\sigma\). Then by Theorem 2.1 \(f^{-1}(A^o) \subseteq (f^{-1}(A))^o\) for every \(A \in \mathcal{I}^Y\). Let \(A \in W(\sigma)\), i.e., \(A = A^o\). Then \(f^{-1}(A) = f^{-1}(A^o) \subseteq (f^{-1}(A))^o\). From the definition of smooth interior we have \((f^{-1}(A))^o \subseteq f^{-1}(A)\). Hence \(f^{-1}(A) = (f^{-1}(A))^o\), i.e., \(f^{-1}(A) \in W(\tau)\). Therefore \(f : X \to Y\) is weak smooth continuous with respect to \(\tau\) and \(\sigma\).

We obtain the following corollary from Theorem 3.4 and 3.5.

Corollary 3.6. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and \(f : X \to Y\) a surjective and quasi-smooth continuous function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is weak smooth compact, then so is \((Y, \sigma)\).

Definition 3.7. A s.t.s. \((X, \tau)\) is called weak smooth nearly compact if for every family \(\{A_i : i \in J\}\) in \(W(\tau)\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\cup_{i \in J_0} (\overline{A_i})^o = 1_X\).
Definition 3.8. A s.t.s. \((X, \tau)\) is called weak smooth almost compact if for every family \(\{A_i : i \in J\}\) in \(W(\tau)\) covering \(X\), there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} A_i = 1_X\).

Theorem 3.9. A weak smooth compact s.t.s. \((X, \tau)\) is weak smooth nearly compact.

Proof. Let \(\{A_i : i \in J\}\) be a family in \(W(\tau)\) covering \(X\). Since \((X, \tau)\) is weak smooth compact, there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} A_i = 1_X\). Since \(A_i \in W(\tau)\) for each \(i \in J\), we have \(A_i = A_i^\circ\) for each \(i \in J\). From Proposition 3.1 \([3]\) we have \(A_i = A_i^\circ \subseteq (\overline{A_i})^\circ\) for each \(i \in J\). Thus \(1_X = \bigcup_{i \in J_0} A_i \subseteq \bigcup_{i \in J_0} (\overline{A_i})^\circ\), i.e., \(\bigcup_{i \in J_0} (\overline{A_i})^\circ = 1_X\). Hence \((X, \tau)\) is weak smooth nearly compact. \(\square\)

Theorem 3.10. A weak smooth nearly compact s.t.s. \((X, \tau)\) is weak smooth almost compact.

Proof. Let \(\{A_i : i \in J\}\) be a family in \(W(\tau)\) covering \(X\). Since \((X, \tau)\) is weak smooth nearly compact, there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} (\overline{A_i})^\circ = 1_X\). Since \((\overline{A_i})^\circ \subseteq \overline{A_i}\) for each \(i \in J\) by Proposition 3.2 \([3]\), \(1_X = \bigcup_{i \in J_0} (\overline{A_i})^\circ \subseteq \bigcup_{i \in J_0} \overline{A_i}\). So \(\bigcup_{i \in J_0} \overline{A_i} = 1_X\). Hence \((X, \tau)\) is weak smooth almost compact. \(\square\)

Theorem 3.11. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and \(f : X \rightarrow Y\) a surjective and quasi-smooth continuous function with respect to \(\tau\) and \(\sigma\). If \((X, \tau)\) is weak smooth almost compact, then so is \((Y, \sigma)\).

Proof. Let \(\{A_i : i \in J\}\) be a family in \(W(\sigma)\) covering \(Y\), i.e., \(\bigcup_{i \in J} A_i = 1_Y\). Then \(1_Y = f^{-1}(1_Y) = \bigcup_{i \in J} f^{-1}(A_i)\). Since \(f\) is quasi-smooth continuous with respect to \(\tau\) and \(\sigma\), \(f\) is weak smooth continuous with respect to \(\tau\) and \(\sigma\) by Theorem 3.5. Hence \(f^{-1}(A_i) \in W(\tau)\) for each \(i \in J\). Since \((X, \tau)\) is weak smooth almost compact, there exists a finite subset \(J_0\) of \(J\) such that \(\bigcup_{i \in J_0} f^{-1}(A_i) = 1_X\). From the surjectivity of \(f\) we have \(1_Y = f(1_X) = f(\bigcup_{i \in J_0} f^{-1}(A_i)) = \bigcup_{i \in J_0} f(f^{-1}(A_i))\). Since \(f : X \rightarrow Y\) is quasi-smooth continuous with respect to \(\tau\) and \(\sigma\), from Theorem 2.1 we have \(f^{-1}(A_i) \subseteq f^{-1}(\overline{A_i})\) for each \(i \in J\). Hence \(1_Y = \bigcup_{i \in J_0} f(f^{-1}(A_i)) \subseteq \bigcup_{i \in J_0} f(f^{-1}(\overline{A_i})) = \bigcup_{i \in J_0} \overline{A_i}\), i.e., \(\bigcup_{i \in J_0} \overline{A_i} = 1_Y\). Thus \((Y, \sigma)\) is weak smooth almost compact. \(\square\)

Theorem 3.12 Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and \(f : X \rightarrow Y\) a surjective, quasi-smooth continuous and smooth
open function with respect to $\tau$ and $\sigma$. If $(X, \tau)$ is weak smooth nearly compact, then so is $(Y, \sigma)$.

**Proof.** Let $\{A_i : i \in J\}$ be a family in $W(\sigma)$ covering $Y$, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since $f$ is quasi-smooth continuous with respect to $\tau$ and $\sigma$, $f$ is weak smooth continuous with respect to $\tau$ and $\sigma$ by Theorem 3.5. Hence $f^{-1}(A_i) \in W(\tau)$ for each $i \in J$. Since $(X, \tau)$ is weak smooth nearly compact, there exists a finite subset $J_0$ of $J$ such that $\cup_{i \in J_0} (f^{-1}(A_i))^o = 1_X$. From the surjectivity of $f$ we have $1_Y = f(1_X) = f(\cup_{i \in J_0}(f^{-1}(A_i))^o) = \cup_{i \in J_0} f((f^{-1}(A_i))^\circ)$. Since $f : X \to Y$ is smooth open with respect to $\tau$ and $\sigma$, from Theorem 2.2 we have $f((f^{-1}(A_i))^\circ) \subseteq (f(f^{-1}(A_i)))^\circ$ for each $i \in J$. Since $f : X \to Y$ is quasi-smooth continuous with respect to $\tau$ and $\sigma$, from Theorem 2.1 we have $\overline{f^{-1}(A_i)} \subseteq f^{-1}(\overline{A_i})$ for each $i \in J$. Hence $1_Y = \cup_{i \in J_0} f((f^{-1}(A_i))^\circ) \subseteq \cup_{i \in J_0} (f(\overline{f^{-1}(A_i)}))^\circ \subseteq \cup_{i \in J_0} (f(\overline{A_i}))^\circ = \cup_{i \in J_0} (\overline{A_i})^\circ$, i.e., $\cup_{i \in J_0} (\overline{A_i})^\circ = 1_Y$. Thus $(Y, \sigma)$ is weak smooth nearly compact. \qed

4. Types of weak smooth $\alpha$-compactness

In this section, we introduce topological concepts of a s.t.s. in terms of the family $W_\alpha(\tau)$ and investigate some of their properties.

**Definition 4.1** [6]. Let $(X, \tau)$ be a s.t.s., $\alpha \in [0, 1)$ and $A \in \mathcal{I}_X$. The $\tau$-smooth $\alpha$-closure (resp., $\tau$-smooth $\alpha$-interior) of $A$, denoted by $\overline{A}_\alpha$ (resp., $A_\alpha^\circ$), is defined by $\overline{A}_\alpha = \cap\{K \in \mathcal{I}_X : \tau^*(K) > \alpha \tau^*(A), A \subseteq K\}$ (resp., $A_\alpha^\circ = \cup\{K \in \mathcal{I}_X : \tau(K) > \alpha \tau(A), K \subseteq A\}$).

We define the families $W_\alpha(\tau) = \{A \in \mathcal{I}_X : A = A_\alpha^\circ\}$ and $W_\alpha^*(\tau) = \{A \in \mathcal{I}_X : A = \overline{A}_\alpha\}$, where $(X, \tau)$ is a s.t.s. Note that $A \in W_\alpha(\tau) \iff A^c \in W_\alpha^*(\tau)$.

**Definition 4.2.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$ if $A \in W_\alpha(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha(\tau)$ for every $A \in \mathcal{I}_Y$.

**Theorem 4.3.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$ if and only if $A \in W_\alpha^*(\sigma) \Rightarrow f^{-1}(A) \in W_\alpha^*(\tau)$ for every $A \in \mathcal{I}_Y$. 
PROOF. The proof follows directly from the definitions of \( W_\alpha(\tau) \), \( W_\alpha^*(\tau) \) and Definition 4.2.

DEFINITION 4.4. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces and let \( \alpha \in [0, 1) \). A function \( f : X \rightarrow Y \) is called weak smooth \( \alpha \)-open (resp., weak smooth \( \alpha \)-closed) with respect to \( \tau \) and \( \sigma \) if \( A \subseteq W_\alpha(\tau) \Rightarrow f(A) \subseteq W_\alpha^*(\sigma) \) (resp., \( A \subseteq W_\alpha^*(\tau) \Rightarrow f(A) \subseteq W_\alpha^*(\sigma) \)) for every \( A \subseteq I^X \).

DEFINITION 4.5. Let \( \alpha \in [0, 1) \). A s.t.s. \((X, \tau)\) is called weak smooth \( \alpha \)-compact if every family in \( W_\alpha(\tau) \) covering \( X \) has a finite subcover.

Note that a weak smooth \( \alpha \)-compact s.t.s. \((X, \tau)\) is smooth compact.

THEOREM 4.6. Let \( \alpha \in [0, 1) \). Then a s.t.s. \((X, \tau)\) is weak smooth \( \alpha \)-compact if and only if every family in \( W_\alpha^*(\tau) \) having the finite intersection property has a non-empty intersection.

PROOF. Let \((X, \tau)\) be a weak smooth \( \alpha \)-compact s.t.s. and let \( \{A_i : i \in J\} \) be a family in \( W_\alpha^*(\tau) \) having the finite intersection property, i.e., for any finite subset \( J_0 \subseteq J \), \( \cap_{i \in J_0} A_i \neq 0_X \). Now suppose that \( \cap_{i \in J} A_i = 0_X \). Then \( \cup_{i \in J} A_i^c = 1_X \). Since \( \{A_i : i \in J\} \subseteq W_\alpha^*(\tau) \), i.e., \( \{A_i^c : i \in J\} \subseteq W_\alpha(\tau) \) and \((X, \tau)\) is a weak smooth \( \alpha \)-compact s.t.s., there exists a finite subset \( J_0 \subseteq J \) such that \( \cup_{i \in J_0} A_i^c = 1_X \). Hence \( \cap_{i \in J_0} A_i = 0_X \), which is a contradiction.

Conversely, suppose that every family in \( W_\alpha^*(\tau) \) having the finite intersection property has a non-empty intersection and \((X, \tau)\) is not a weak smooth \( \alpha \)-compact s.t.s. Then there exists a family \( \{A_i : i \in J\} \) in \( W_\alpha(\tau) \) covering \( X \) such that for any finite subset \( J_0 \subseteq J \), \( \cup_{i \in J_0} A_i \neq 1_X \), i.e., \( \cap_{i \in J_0} A_i^c \neq 0_X \). Since \( \{A_i : i \in J\} \subseteq W_\alpha(\tau) \), we have \( \{A_i^c : i \in J\} \subseteq W_\alpha^*(\tau) \). Hence the family \( \{A_i^c : i \in J\} \) has the finite intersection property. From the hypothesis we have \( \cap_{i \in J} A_i^c \neq 0_X \). Hence \( \cup_{i \in J} A_i \neq 1_X \), which is a contradiction.

THEOREM 4.7. Let \((X, \tau)\) and \((Y, \sigma)\) be two smooth topological spaces, \( \alpha \in [0, 1) \) and \( f : X \rightarrow Y \) a surjective and weak smooth \( \alpha \)-continuous function with respect to \( \tau \) and \( \sigma \). If \((X, \tau)\) is weak smooth \( \alpha \)-compact, then so is \((Y, \sigma)\).

PROOF. Let \( \{A_i : i \in J\} \) be a family in \( W_\alpha(\sigma) \) covering \( Y \), i.e., \( \cup_{i \in J} A_i = 1_Y \). Then \( \cup_{i \in J} f^{-1}(A_i) = f^{-1}(1_Y) = 1_X \). Since \( f : X \rightarrow Y \) is weak smooth \( \alpha \)-continuous with respect to \( \tau \) and \( \sigma \), \( \{f^{-1}(A_i) : i \in J\} \subseteq W_\alpha^*(\tau) \). Since \((X, \tau)\) is weak smooth \( \alpha \)-compact, there exists a finite
subset $J_0 \subseteq J$ such that $\bigcup_{i \in J_0} f^{-1}(A_i) = 1_X$. From the surjectivity of $f$ we have $1_Y = f(1_X) = f(\bigcup_{i \in J_0} f^{-1}(A_i)) = \bigcup_{i \in J_0} f(f^{-1}(A_i)) = \bigcup_{i \in J_0} A_i$. Therefore $(Y, \sigma)$ is weak smooth $\alpha$-compact. \hfill $\square$

**Definition 4.8** [6]. Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0, 1)$. A function $f : X \to Y$ is called smooth $\alpha$-preserving (resp., strict smooth $\alpha$-preserving) with respect to $\tau$ and $\sigma$ if $\sigma(A) \geq \alpha \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) \geq \alpha \tau(f^{-1}(B))$ (resp., $\sigma(A) > \alpha \sigma(B) \Leftrightarrow \tau(f^{-1}(A)) > \alpha \tau(f^{-1}(B))$) for every $A, B \in I_Y$.

A function $f : X \to Y$ is called smooth open $\alpha$-preserving (resp., strict smooth open $\alpha$-preserving) with respect to $\tau$ and $\sigma$ if $\tau(A) \geq \alpha \tau(B) \Rightarrow \sigma(f(A)) \geq \alpha \sigma(f(B))$ (resp., $\tau(A) > \alpha \tau(B) \Rightarrow \sigma(f(A)) > \alpha \sigma(f(B))$) for every $A, B \in I_Y$.

**Theorem 4.9.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces and let $\alpha \in [0, 1)$. If a function $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, then $f : X \to Y$ is weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$.

**Proof.** Let $f : X \to Y$ be a strict smooth $\alpha$-preserving function with respect to $\tau$ and $\sigma$. Then by Theorem 3.13 [6] $f^{-1}(A^\alpha_\sigma) \subseteq (f^{-1}(A))^{\alpha}_\sigma$ for every $A \in I_Y$. Let $A \in W_\alpha(\sigma)$, i.e., $A = A^\alpha_\sigma$. Then by the above result $f^{-1}(A) = f^{-1}(A^\alpha_\sigma) \subseteq (f^{-1}(A))^{\alpha}_\sigma$. From Theorem 3.5 [6] we have $f^{-1}(A) = (f^{-1}(A))^{\alpha}_\sigma$, i.e., $f^{-1}(A) \in W_\alpha(\tau)$. Therefore $f : X \to Y$ is weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$. \hfill $\square$

We obtain the following corollary from Theorem 4.7 and 4.9.

**Corollary 4.10.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and strict smooth $\alpha$-preserving function with respect to $\tau$ and $\sigma$. If $(X, \tau)$ is weak smooth $\alpha$-compact, then so is $(Y, \sigma)$.

**Definition 4.11.** Let $\alpha \in [0, 1)$. A s.t.s. $(X, \tau)$ is called weak smooth nearly $\alpha$-compact if for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering $X$, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0}((A_i)_\alpha^{\alpha}) = 1_X$.

**Definition 4.12.** Let $\alpha \in [0, 1)$. A s.t.s. $(X, \tau)$ is called weak smooth almost $\alpha$-compact if for every family $\{A_i : i \in J\}$ in $W_\alpha(\tau)$ covering $X$, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0}A_i = 1_X$. 

150 Chun-Kee Park, Won Keun Min and Myeong Hwan Kim
A smooth topology $\tau : I^X \to I$ on $X$ is called monotonic increasing (resp., monotonic decreasing) if $A \subseteq B \Rightarrow \tau(A) \leq \tau(B)$ (resp., $A \subseteq B \Rightarrow \tau(A) \geq \tau(B)$) for every $A, B \in I^X$ [6].

**Theorem 4.13.** Let $(X, \tau)$ be a s.t.s., $\alpha \in [0, 1)$ and $\tau$ a monotonic decreasing smooth topology. If $(X, \tau)$ is weak smooth $\alpha$-compact, then $(X, \tau)$ is weak smooth nearly $\alpha$-compact.

**Proof.** Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\tau)$ covering $X$. Since $(X, \tau)$ is weak smooth $\alpha$-compact, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0} A_i = 1_X$. Since $A_i \in W_\alpha(\tau)$ for each $i \in J$, $A_i = (A_i}_\alpha$ for each $i \in J$. Since $\tau$ is monotonic decreasing and $A_i \subseteq (A_i}_\alpha$ for each $i \in J$, $\tau(A_i) \geq \tau((A_i}_\alpha)$ for each $i \in J$. Hence from Theorem 3.2 [6] we have $A_i = (A_i}_\alpha \subseteq (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha$ for each $i \in J$. Thus $1_X = \bigcup_{i \in J_0} A_i \subseteq \bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha$, i.e., $\bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha = 1_X$. Hence $(X, \tau)$ is weak smooth nearly $\alpha$-compact. □

**Theorem 4.14.** Let $\alpha \in [0, 1)$. Then a weak smooth nearly $\alpha$-compact s.t.s. $(X, \tau)$ is weak smooth almost $\alpha$-compact.

**Proof.** Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\tau)$ covering $X$. Since $(X, \tau)$ is weak smooth nearly $\alpha$-compact, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha = 1_X$. Since $\bigcap_{i \in J_0}(A_i}_\alpha \subseteq (A_i}_\alpha$ for each $i \in J$ by Theorem 3.5 [6], $1_X = \bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha \subseteq \bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha$. So $\bigcup_{i \in J_0} (\bigcap_{i \in J_0}(A_i}_\alpha)^\alpha = 1_X$. Hence $(X, \tau)$ is weak smooth almost $\alpha$-compact. □

**Theorem 4.15.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective and strict smooth $\alpha$-preserving function with respect to $\tau$ and $\sigma$. If $(X, \tau)$ is weak smooth almost $\alpha$-compact, then so is $(Y, \sigma)$.

**Proof.** Let $\{A_i : i \in I\}$ be a family in $W_\alpha(\sigma)$ covering $Y$, i.e., $\bigcup_{i \in I} A_i = 1_Y$. Then $1_Y = f^{-1}(1_Y) = \bigcup_{i \in J} f^{-1}(A_i)$. Since $f$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, $f$ is weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$ by Theorem 4.9. Hence $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since $(X, \tau)$ is weak smooth almost $\alpha$-compact, there exists a finite subset $J_0$ of $J$ such that $\bigcup_{i \in J_0} (f^{-1}(A_i))_\alpha = 1_X$. From the surjectivity of $f$ we have $1_Y = f(1_X) = f(\bigcup_{i \in J_0} (f^{-1}(A_i))_\alpha) = \bigcup_{i \in J_0} f((f^{-1}(A_i))_\alpha)$. Since $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, from Theorem 3.13 [6] we have $(f^{-1}(A_i))_\alpha \subseteq
$f^{-1}(\overline{A_i}_\alpha)$ for each $i \in J$. Hence

$$1_Y = \cup_{i \in J_0} f((f^{-1}(A_i))_\alpha) \subseteq \cup_{i \in J_0} f(f^{-1}(A_i)) = \cup_{i \in J_0} \overline{A_i}_\alpha,$$

i.e., $\cup_{i \in J_0} \overline{A_i}_\alpha = 1_Y$. Thus $(Y, \sigma)$ is weak smooth almost $\alpha$-compact. \qed

**Theorem 4.16.** Let $(X, \tau)$ and $(Y, \sigma)$ be two smooth topological spaces, $\alpha \in [0, 1)$ and $f : X \to Y$ a surjective, strict smooth $\alpha$-preserving and smooth strict open $\alpha$-preserving function with respect to $\tau$ and $\sigma$. If $(X, \tau)$ is weak smooth nearly $\alpha$-compact, then so is $(Y, \sigma)$. 

**Proof.** Let $\{A_i : i \in J\}$ be a family in $W_\alpha(\sigma)$ covering $Y$, i.e., $\cup_{i \in J} A_i = 1_Y$. Then $1_X = f^{-1}(1_Y) = \cup_{i \in J} f^{-1}(A_i)$. Since $f$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, $f$ is weak smooth $\alpha$-continuous with respect to $\tau$ and $\sigma$ by Theorem 4.9. Hence $f^{-1}(A_i) \in W_\alpha(\tau)$ for each $i \in J$. Since $(X, \tau)$ is weak smooth nearly $\alpha$-compact, there exists a finite subset $J_0$ of $J$ such that $\cup_{i \in J_0} ((f^{-1}(A_i))_\alpha)_\alpha = 1_X$. From the surjectivity of $f$ we have $1_Y = f(1_X) = f(\cup_{i \in J_0} ((f^{-1}(A_i))_\alpha)_\alpha) = \cup_{i \in J_0} f((f^{-1}(A_i))_\alpha)_\alpha).$ Since $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, from Theorem 3.14 [6] we have $f((f^{-1}(A_i))_\alpha)_\alpha) \subseteq (f((f^{-1}(A_i))_\alpha)_\alpha)$ for each $i \in J$. Since $f : X \to Y$ is strict smooth $\alpha$-preserving with respect to $\tau$ and $\sigma$, from Theorem 3.13 [6] we have $((f^{-1}(A_i))_\alpha) \subseteq f^{-1}(\overline{A_i}_\alpha)$ for each $i \in J$. Hence

$$1_Y = \cup_{i \in J_0} f((f^{-1}(A_i))_\alpha)_\alpha) \subseteq \cup_{i \in J_0} f((f^{-1}(A_i))_\alpha)_\alpha) \subseteq \cup_{i \in J_0} \overline{A_i}_\alpha$$

i.e.,

$$\cup_{i \in J_0} \overline{A_i}_\alpha = 1_Y.$$

Thus $(Y, \sigma)$ is weak smooth nearly $\alpha$-compact. \qed

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