A CONVERGENCE THEOREM FOR FEYNMAN’S OPERATIONAL CALCULUS: THE CASE OF TIME DEPENDENT NONCOMMUTING OPERATORS

BYUNG MOO ANH* AND CHOON HO LEE

Abstract. Feynman’s operational calculus for noncommuting operators was studied via measures on the time interval. We investigate that if a sequence of \( p \)-tuples of measures converges to another \( p \)-tuple of measures, then the corresponding sequence of operational calculi in the time dependent setting converges to the operational calculus determined by the limiting \( p \)-tuple of measures.

1. Introduction

Let \( X \) be a separable Banach space over the complex numbers and let \( \mathcal{L}(X) \) denote the space of bounded linear operators on \( X \). Fix \( T > 0 \). For \( i = 1, \ldots, p \) let \( A_i : [0,T] \to \mathcal{L}(X) \) be maps that are measurable in the sense that \( A_i^{-1}(E) \) is a Borel set in \([0,T]\) for any strong operator open set \( E \subset \mathcal{L}(X) \). To each \( A_i(\cdot) \) we associate a finite continuous Borel measure \( \mu_i \) on \([0,T]\) and we require that, for each \( i \),

\[
  r_i = \int_{[0,T]} ||A_i(s)||\mathcal{L}(X)|\mu_i|(ds) < \infty.
\]

Given a positive integer \( p \) and \( p \) positive numbers \( r_1, \ldots, r_p \), let \( \mathcal{A}(r_1, \ldots, r_p) \) be the space of complex-valued functions of \( p \) complex variables \( f(z_1, \ldots, z_p) \), which are analytic at \((0, \cdots, 0)\), and are such that their power series expansion

\[
  f(z_1, \cdots, z_p) = \sum_{m_1, \cdots, m_p=0}^{\infty} c_{m_1, \cdots, m_p} z_1^{m_1} \cdots z_p^{m_p}
\]

Received February 5, 2004.
2000 Mathematics Subject Classification: 47A60.
Key words and phrases: Feynman’s operational calculus, disentangling.
*This work was supported by Korea Research Foundation Grant (KRF-2001-002-D00035).
converges absolutely, at least on the closed polydisk \(|z_1| \leq r_1, \cdots, |z_p| \leq r_p\). Such functions are analytic at least in the open polydisk \(|z_1| < r_1, \cdots, |z_p| < r_p\).

For \(f \in \mathcal{A}(r_1, \cdots, r_p)\) given by (1), we let

\[
\|f\| = \|f\|_{\mathcal{A}(r_1, \cdots, r_p)} := \sum_{m_1, \cdots, m_p=0}^{\infty} |c_{m_1, \cdots, m_p}| r_1^{m_1} \cdots r_p^{m_p}.
\]

The function on \(\mathcal{A}(r_1, \cdots, r_p)\) defined by (2) makes \(\mathcal{A}(r_1, \cdots, r_p)\) into a commutative Banach algebra [3].

To the algebra \(\mathcal{A}(r_1, \cdots, r_p)\) we associate a disentangling algebra by replacing the \(z_i\)'s with formal commuting objects \((A_i(\cdot), \mu_i)\), \(i = 1, \cdots, p\). Consider the collection \(\mathbb{D}(\{(A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p)\})\) of all expressions of the form

\[
f((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p)) = \sum_{m_1, \cdots, m_p=0}^{\infty} c_{m_1, \cdots, m_p} ((A_1(\cdot), \mu_1))^{m_1} \cdots ((A_p(\cdot), \mu_p))^{m_p}
\]

where \(c_{m_1, \cdots, m_p} \in \mathbb{C}\) for all \(m_1, \cdots, m_p = 0, 1, \cdots, \)

\[
\|f((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))\| = \|f((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))\|_{\mathbb{D}((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))} := \sum_{m_1, \cdots, m_p=0}^{\infty} |c_{m_1, \cdots, m_p}| r_1^{m_1} \cdots r_p^{m_p} < \infty
\]

where \(r_i = \int_{0,2} ||A_i(s)||_{\mathcal{L}(\mathcal{X})} |\mu_i||\,ds\), \(i = 1, 2, \cdots, p\).

Rather than using the notation \((A_i(\cdot), \mu_i)\) below, we will often abbreviate to \(A_i(\cdot)\), especially when carrying out calculations. We will often write \(\mathbb{D}\) in place of \(\mathbb{D}(\{A_1(\cdot), \mu_1\}, \cdots, A_p(\cdot))\) or \(\mathbb{D}(\{A_1(\cdot), \mu_1\}, \cdots, (A_p(\cdot), \mu_p)\})\).

Adding and scalar multiplying such expressions coordinatewise, we can easily see that \(\mathbb{D}(\{A_1(\cdot), \mu_1\}, \cdots, (A_p(\cdot), \mu_p)\})\) is a vector space and that \(\| \cdot \|_\mathbb{D}\) defined by (3) is a norm. The normed linear space \(\mathbb{D}(\{A_1(\cdot), \mu_1\}, \cdots, (A_p(\cdot), \mu_p)\}), \| \cdot \|_\mathbb{D}\) can be identified with the weighted \(l_1\)-space, where the weight at the index \((m_1, \cdots, m_p)\) is \(r_1^{m_1} \cdots r_p^{m_p} \). It follows that \(\mathbb{D}(\{A_1(\cdot), \mu_1\}, \cdots, (A_p(\cdot), \mu_p)\})\) is a commutative Banach algebra with identity [7].
We refer to $\mathcal{D}((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))$ as the disentangling algebra associated with the $p$-tuple $((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))$.

For $m = 0, 1, \cdots$, let $S_m$ denote the set of all permutations of the integers $\{1, \cdots, m\}$, and given $\pi \in S_m$ we let

$$\Delta_m(\pi) = \{(s_1, \cdots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\}.$$ 

Now for nonnegative integers $m_1, \cdots, m_p$ and $m = m_1 + \cdots + m_p$, we define

$$C_i(s) = \begin{cases} A_1(s), & \text{if } i \in \{1, \cdots, m_1\} \\ A_2(s), & \text{if } i \in \{m_1 + 1, \cdots, m_1 + m_2\} \\ \vdots & \\ A_p(s), & \text{if } i \in \{m_1 + \cdots + m_{p-1} + 1, \cdots, m\} \end{cases}$$

for $i = 1, \cdots, m$ and for all $0 \leq s \leq T$.

**Definition 1.** Let $P_{m_1, \cdots, m_p}(z_1, \cdots, z_p) = z_1^{m_1} \cdots z_p^{m_p}$. We define the action of the disentangling map on this monomial by

$$T_{\mu_1, \cdots, \mu_p}P_{m_1, \cdots, m_p}(A_1(\cdot), \cdots, A_p(\cdot)) = T_{\mu_1, \cdots, \mu_p}((A_1(\cdot))^{m_1} \cdots (A_p(\cdot))^{m_p})$$

$$:= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})$$

$$(\mu_1^{m_1} \times \cdots \times \mu_p^{m_p})(ds_1, \cdots, ds_m).$$

Finally for $f \in \mathcal{D}((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p))$ given by

$$f(A_1(\cdot), \cdots, A_p(\cdot)) = \sum_{m_1, \cdots, m_p = 0}^{\infty} c_{m_1, \cdots, m_p}(A_1(\cdot))^{m_1} \cdots (A_p(\cdot))^{m_p}$$

we set

$$T_{\mu_1, \cdots, \mu_p}f(A_1(\cdot), \cdots, A_p(\cdot))$$

$$:= \sum_{m_1, \cdots, m_p = 0}^{\infty} c_{m_1, \cdots, m_p}(A_1(\cdot))^{m_1} \cdots (A_p(\cdot))^{m_p}.$$

We will often use the alternative notation:

$$P_{\mu_1, \cdots, \mu_p}(A_1(\cdot), \cdots, A_p(\cdot)) = T_{\mu_1, \cdots, \mu_p}P_{m_1, \cdots, m_p}(A_1(\cdot), \cdots, A_p(\cdot))$$

and

$$f_{\mu_1, \cdots, \mu_p}(A_1(\cdot), \cdots, A_p(\cdot)) = T_{\mu_1, \cdots, \mu_p}f(A_1(\cdot), \cdots, A_p(\cdot)).$$

The following result is Proposition 2.2 of [7].
Proposition 1. The disentangling map $T_{\mu_1, \ldots, \mu_p}$ is a bounded linear operator from $D((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p))$ to $L(X)$. In fact, $\|T_{\mu_1, \ldots, \mu_p}\| \leq 1$.

2. Stability theorem

Let $\{\nu_n\}_{n=1}^\infty$ be a sequence of Borel probability measures on $[0, T]$. We say that $\nu_n$ converges weakly to a Borel probability measure $\nu$ and write $\nu_n \rightharpoonup \nu$ provided that

$$\int_{[0, T]} b(s) \nu_n(ds) \to \int_{[0, T]} b(s) \nu(ds)$$

for every bounded continuous function $b$ on $[0, T]$.

Proposition 2. Let $A_i : [0, T] \to L(X)$ be continuous for each $i = 1, 2, \ldots, p$. Let $\{\mu_i,n\}_{n=1}^\infty$ be sequences of continuous Borel probability measures on $[0, T]$ such that $\mu_{i,n} \rightharpoonup \mu_i$ for each $i$. Then for any nonnegative integers $m_1, \ldots, m_p$ and for any $\phi \in X$

$$\lim_{n \to \infty} P^{m_1, \ldots, m_p}_{\mu_1,n, \ldots, \mu_p,n}(A_1(\cdot), \ldots, A_p(\cdot))\phi = P^{m_1, \ldots, m_p}_{\mu_1, \ldots, \mu_p}(A_1(\cdot), \ldots, A_p(\cdot))\phi.$$

Proof. $\{\mu_{i,n}^{m_1} \times \cdots \times \mu_{i,n}^{m_p}\}$ is a sequence of continuous probability measures on $[0, T]^m$ since each term in the product is a continuous probability measure. And $[0, T]^m$ is separable. By Theorem 3.2 of [1] $\mu_{i,n}^{m_1} \times \cdots \times \mu_{i,n}^{m_p} \rightharpoonup \mu_{i,n}^{m_1} \times \cdots \times \mu_{i,n}^{m_p}$ since $\mu_{i,n} \rightharpoonup \mu_i$ for each $i$. For each $\phi \in X$, $C_{\pi(m)}(\cdot) \cdots C_{\pi(1)}(\cdot) : [0, T]^m \to X$ is continuous for each $\pi \in S_m$. From Theorem 5.1 of [1] we have

$$\lim_{n \to \infty} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})\phi$$

$$= \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)})\phi$$

Hence the conclusion follows. $\square$
Lemma 3. Let \( \mu_1, \ldots, \mu_p, \mu_1, \ldots, \mu_p, n = 1, 2, \ldots \) be continuous probability measures. Suppose for \( i = 1, 2, \ldots, p \)

\[
\bar{r}_i = \sup \{ r_{i,1}, \ldots, r_{i,n}, \cdots \} < \infty
\]

where \( r_i = \int_{[0,T]} \| A_i(s) \| |\mu_i|(ds) \) and \( r_{i,n} = \int_{[0,T]} \| A_i(s) \| |\mu_{i,n}|(ds) \).

Then for any \( f \in \mathcal{A}(\bar{r}_1, \ldots, \bar{r}_p) \),

\[
f((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)) \in \mathbb{D}(\mathcal{D}_1(\cdot), \mu_1, \ldots, \mathcal{D}_p(\cdot), \mu_p)
\]

and

\[
f((A_1(\cdot), \mu_1, n), \ldots, (A_p(\cdot), \mu_p, n)) \in \mathbb{D}(\mathcal{D}_1(\cdot), \mu_1, n, \ldots, \mathcal{D}_p(\cdot), \mu_p, n)
\]

for any \( n = 1, 2, \ldots \).

Proof. Suppose that

\[
f(z_1, \ldots, z_p) = \sum_{m_1, \ldots, m_p = 0}^{\infty} c_{m_1, \ldots, m_p} z_1^{m_1} \cdots z_p^{m_p}
\]

such that \( \sum_{m_1, \ldots, m_p = 0}^{\infty} |c_{m_1, \ldots, m_p}| \bar{r}_1^{m_1} \cdots \bar{r}_p^{m_p} < \infty \). Then

\[
||f((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p))|| = \sum_{m_1, \ldots, m_p = 0}^{\infty} |c_{m_1, \ldots, m_p}| \left[ \int_{[0,T]} \| A_1(s) \| |\mu_1|(ds) \right]^{m_1} \cdots \left[ \int_{[0,T]} \| A_p(s) \| |\mu_p|(ds) \right]^{m_p}
\]

\[
\leq \sum_{m_1, \ldots, m_p = 0}^{\infty} |c_{m_1, \ldots, m_p}| \bar{r}_1^{m_1} \cdots \bar{r}_p^{m_p} < \infty.
\]

Hence \( f((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)) \in \mathbb{D}(\mathcal{D}_1(\cdot), \mu_1, \ldots, \mathcal{D}_p(\cdot), \mu_p) \).

Similarly we can check that \( f((A_1(\cdot), \mu_1, n), \ldots, (A_p(\cdot), \mu_p, n)) \in \mathbb{D}(\mathcal{D}_1(\cdot), \mu_1, n, \ldots, \mathcal{D}_p(\cdot), \mu_p, n) \). \( \square \)
**Theorem 4.** Let the hypotheses of Proposition 2 be satisfied. Further suppose that for each $i = 1, 2, \cdots, p$ and $n = 1, 2, \cdots, r_i, r_{i,n}$ are given as in Lemma 3. Let $T_{\mu_{1,n}, \cdots, \mu_{p,n}}$ denote the disentangling map corresponding to the $n^\text{th}$ term of sequences of measures. Then for any $f \in \mathcal{A}(\bar{r}_1, \cdots, \bar{r}_p)$, and for any $\phi \in X$,

$$
\lim_{n \to \infty} T_{\mu_{1,n}, \cdots, \mu_{p,n}} f((A_1(\cdot), \mu_{1,n}), \cdots, (A_p(\cdot), \mu_{p,n})) \phi = T_{\mu_1, \cdots, \mu_p} f((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p)) \phi.
$$

**Proof.** We have

$$
\|T_{\mu_{1,n}, \cdots, \mu_{p,n}} f((A_1(\cdot), \mu_{1,n}), \cdots, (A_p(\cdot), \mu_{p,n})) \phi - T_{\mu_1, \cdots, \mu_p} f((A_1(\cdot), \mu_1), \cdots, (A_p(\cdot), \mu_p)) \phi\|

\leq \sum_{m_1, \cdots, m_p=0}^{\infty} |c_{m_1, \cdots, m_p}| |P_{\mu_{1,n}, \cdots, \mu_{p,n}}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot)) \phi |

- P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot)) \phi ||.
$$

Note that

$$
|c_{m_1, \cdots, m_p}| |P_{\mu_{1,n}, \cdots, \mu_{p,n}}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot)) \phi |

- P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot)) \phi ||

\leq |c_{m_1, \cdots, m_p}| ||P_{\mu_{1,n}, \cdots, \mu_{p,n}}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot))||

+ ||P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p} (A_1(\cdot), \cdots, A_p(\cdot))|| ||\phi||

\leq |c_{m_1, \cdots, m_p}| ||\int_{[0,T]} |A_1(s)|| \mu_{1,n}(\mid (ds)\mid |^{m_1} \cdots

\int_{[0,T]} |A_p(s)|| \mu_{p,n}(\mid (ds)\mid |^{m_p} + \int_{[0,T]} ||A_1(s)|| \mu_{1}(\mid (ds)\mid |^{m_1} \cdots

\int_{[0,T]} ||A_p(s)|| \mu_{p}(\mid (ds)\mid |^{m_p}) ||\phi||

= |c_{m_1, \cdots, m_p}| [r_{1,n}^{m_1} \cdots r_{p,n}^{m_p} + r_{1}^{m_1} \cdots r_{p}^{m_p}] ||\phi||

\leq 2|c_{m_1, \cdots, m_p}| r_{1,n}^{m_1} \cdots r_{p,n}^{m_p} ||\phi||.
Then for any 

Further assume that 

Lebesgue Dominated Convergence Theorem, we obtain the result. □

**Theorem 5.** Let $A_i : [0, T] \to \mathcal{L}(X)$ be measurable for each $i = 1, 2, \cdots, p$. Let \( \{\mu_{i,n}\}_{n=1}^{\infty} \) for $i = 1, 2, \cdots, p$ be sequences of continuous Borel probability measures on $[0, T]$ such that $\mu_{i,n} \to \mu_i$ for each $i$. Further assume that $M_i := \sup_{s \in [0,T]} |A_i(s)| < \infty$ for each $i = 1, \cdots, p$. Then for any $f \in \mathcal{A}(M_1, \cdots, M_p)$,

\[
\lim_{n \to \infty} T_{\mu_{1,n}} \cdots \mu_{p,n} f((A_1(\cdot), \mu_{1,n}^\top), \cdots, (A_p(\cdot), \mu_{p,n}^\top)) = T_{\mu_1} \cdots \mu_p f((A_1(\cdot), \mu_1^\top), \cdots, (A_p(\cdot), \mu_p^\top)).
\]

**Proof.** First we consider $P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p}(A_1(\cdot), \cdots, A_p(\cdot))$. We see that

\[
||T_{\mu_{1,n}} \cdots \mu_{p,n} P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p}(A_1(\cdot), \mu_{1,n}^\top), \cdots, (A_p(\cdot), \mu_{p,n}^\top))
\]

\[
- T_{\mu_1} \cdots \mu_p P_{\mu_1, \cdots, \mu_p}^{m_1, \cdots, m_p}(A_1(\cdot), \mu_1^\top), \cdots, (A_p(\cdot), \mu_p^\top))||
\]

\[
= || \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p})
\]

\[
(\mu_{1,n}^{\top}) \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{m_1} \times \cdots \mu_p^{m_p})
\]

\[
(\mu_1^{\top}) \leq \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \left| C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \right|
\]

\[
|\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p}(ds_1, \cdots, ds_m) - \mu_1^{m_1} \times \cdots \times \mu_p^{m_p}(ds_1, \cdots, ds_m)|
\]

\[
\leq \sum_{\pi \in S_m} M_1^{m_1} \cdots M_p^{m_p} |\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p}(\Delta_m(\pi)) -
\]

\[
\mu_1^{m_1} \times \cdots \times \mu_p^{m_p}(\Delta_m(\pi))|.
\]

Here $\{\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p}\}$ is a sequence of continuous probability measures on $[0, T]^m$. Since $[0, T]^m$ is separable and $\mu_{i,n} \to \mu_i$ for each $i$, $\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p} \to \mu_1^{m_1} \times \cdots \times \mu_p^{m_p}$ using Theorem 3.2 of [1]. We can apply (v) of Theorem 2.1 of [1] to conclude that

\[
|\mu_{1,n}^{m_1} \times \cdots \times \mu_{p,n}^{m_p}(\Delta_m(\pi)) - \mu_1^{m_1} \times \cdots \times \mu_p^{m_p}(\Delta_m(\pi))| \to 0
\]
as \( n \to \infty \). We therefore conclude

\[
\lim_{n \to \infty} T_{\mu_1, \ldots, \mu_p, n} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n})) = T_{\mu_1, \ldots, \mu_p} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)).
\]

We now turn to \( T_{\mu_1, \ldots, \mu_p} f((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)) \). For \( f \in \mathcal{A}(M_1, \ldots, M_p) \) we have

\[
\| T_{\mu_1, \ldots, \mu_p, n} f((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n})) - T_{\mu_1, \ldots, \mu_p} f((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)) \|
\]

\[
\leq \sum_{m_1, \ldots, m_p=0}^{\infty} c_{m_1, \ldots, m_p} \| T_{\mu_1, \ldots, \mu_p, n} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n})
\]

\[
-(A_p(\cdot), \mu_{p,n}))
\]

\[
\leq \sum_{m_1, \ldots, m_p=0}^{\infty} |c_{m_1, \ldots, m_p}| \| T_{\mu_1, \ldots, \mu_p, n} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n}))
\]

\[
-(A_p(\cdot), \mu_{p,n}))
\]

Now

\[
|c_{m_1, \ldots, m_p}| \| T_{\mu_1, \ldots, \mu_p, n} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n}))
\]

\[
-(A_p(\cdot), \mu_{p,n}))
\]

\[
\leq |c_{m_1, \ldots, m_p}| \left( \| T_{\mu_1, \ldots, \mu_p, n} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_{1,n}), \ldots, (A_p(\cdot), \mu_{p,n})) \| + \| T_{\mu_1, \ldots, \mu_p} P^{m_1, \ldots, m_p}((A_1(\cdot), \mu_1), \ldots, (A_p(\cdot), \mu_p)) \| \right)
\]

\[
\leq |c_{m_1, \ldots, m_p}| \left( \left[ \int_{[0,T]} \| A_1(s) \| \mu_{1,n}((ds))^{m_1} \right] \cdots \left[ \int_{[0,T]} \| A_p(s) \| \mu_{p,n}((ds))^{m_p} \right] \
\right.
\]

\[
+ \left. \left[ \int_{[0,T]} \| A_1(s) \| \mu_1((ds))^{m_1} \right] \cdots \left[ \int_{[0,T]} \| A_p(s) \| \mu_p((ds))^{m_p} \right] \right)
\]

\[
\leq 2 |c_{m_1, \ldots, m_p}| M_1^{m_1} \cdots M_p^{m_p}.
\]

Since \( \sum_{m_1, \ldots, m_p=0}^{\infty} |c_{m_1, \ldots, m_p}| M_1^{m_1} \cdots M_p^{m_p} < \infty \), by (4) and Lebesgue Dominated Convergence Theorem, we obtain the result. \( \square \)
Corollary 6. Assume the same hypotheses as in Theorem 5. Then for any \( f \in \mathcal{A}(M_1, \cdots, M_p) \), and for any \( \phi \in X \),

\[
\lim_{n \to \infty} T_{\mu_1, \cdots, \mu_p, n} f((A_1, \mu_1, n), \cdots, (A_p, \mu_p)) \phi
= T_{\mu_1, \cdots, \mu_p} f((A_1, \mu_1), \cdots, (A_p, \mu_p)) \phi.
\]

Proof. Let \( f \in \mathcal{A}(M_1, \cdots, M_p) \). Then for any \( \phi \in X \), we have

\[
||T_{\mu_1, \cdots, \mu_p, n} f((A_1, \mu_1, n), \cdots, (A_p, \mu_p, n)) \phi
- T_{\mu_1, \cdots, \mu_p} f((A_1, \mu_1), \cdots, (A_p, \mu_p)) \phi||
\leq ||T_{\mu_1, \cdots, \mu_p, n} f((A_1, \mu_1, n), \cdots, (A_p, \mu_p, n))
- T_{\mu_1, \cdots, \mu_p} f((A_1, \mu_1), \cdots, (A_p, \mu_p)) || \phi || \to 0
\]
as \( n \to \infty \) by Theorem 5. This finishes the proof. \( \square \)

References

Byung Moo Ahn
Department of Mathematics
Soonchynhyang University Asan
Chungnam 336-745, Korea
E-mail: anbymo@sch.ac.kr

Choon Ho Lee
Department of Mathematics
Hoseo University Asan
Chungnam 336-795, Korea
E-mail: chlee@math.hoseo.ac.kr