INTUITIONISTIC FUZZY SEMI-PREOPEN SETS AND
INTUITIONISTIC FUZZY SEMI-PRECONTINUOUS MAPPINGS

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Abstract. Using the notion of intuitionistic fuzzy sets, the concept of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings are introduced. The relation between an intuitionistic fuzzy precontinuous mapping and an intuitionistic semi-precontinuous mapping is given. Characterizations of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings are given.

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1. Introduction

After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov is one among them. Using the notion of intuitionistic fuzzy sets, Çoker [4] introduced the notion of intuitionistic fuzzy topological spaces. In this paper, we define the notion of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings. We give relations between an intuitionistic fuzzy precontinuous mapping and an intuitionistic fuzzy semi-precontinuous mapping. We provide characterizations of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings.
2. Preliminaries

Definition 2.1. (Atanassov [1]) An intuitionistic fuzzy set (IFS for short) $A$ in $X$ is an object having the form

$$A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in X \}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Denote by $IFS(X)$ the set of all intuitionistic fuzzy sets in $X$.

Definition 2.2. (Atanassov [1]) Let $A$ and $B$ be IFSs of the form

$$A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in X \} \text{ and } B = \{ (x, \mu_B(x), \gamma_B(x)) \mid x \in X \}.$$ 

Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
(c) $A \land B = \{ (x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) \mid x \in X \}$,
(d) $A \lor B = \{ (x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x)) \mid x \in X \}$.

For the sake of simplicity, we shall use the notation $A = \langle x, \mu_A, \gamma_A \rangle$ instead of $A = \{ (x, \mu_A(x), \gamma_A(x)) \mid x \in X \}$. A constant fuzzy set taking value $\alpha \in [0,1]$ will be denoted by $\alpha$. The IFSs $0_\sim$ and $1_\sim$ are defined to be $0_\sim = \langle x, 0, 1 \rangle$ and $1_\sim = \langle x, 1, 0 \rangle$, respectively. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP for short), written $p_{(\alpha,\beta)}$, is defined to be an IFS of $X$ given by

$$p_{(\alpha,\beta)}(x) := \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0, 1) & \text{otherwise.} \end{cases}$$

Let $f$ be a mapping from a set $X$ to a set $Y$. If

$$B = \{ (y, \mu_B(y), \gamma_B(y)) : y \in Y \}$$

is an IFS in $Y$, then the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is the IFS in $X$ defined by

$$f^{-1}(B) = \{ (x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)) : x \in X \}.$$ 

Çoker [4] generalized the concept of fuzzy topological space, first initiated by Chang [3], to the case of intuitionistic fuzzy sets as follows:

Definition 2.3. ([4] Definition 3.1) An intuitionistic fuzzy topology (IFT for short) on $X$ is a family $\tau$ of IFSs in $X$ satisfying the following axioms:

(T1) $0_\sim, 1_\sim \in \tau$,
(T2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$, 

(T3) \( \cup G_i \in \tau \) for any family \( \{ G_i \mid i \in J \} \subseteq \tau \).

In this case the pair \((X, \tau)\) is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS for short) in \( X \). The complement \( \bar{A} \) of an IFOS \( A \) in IFTS \((X, \tau)\) is called an intuitionistic fuzzy closed set (IFCS for short) in \( X \).

**Definition 2.4.** ([4] Definition 3.13) Let \((X, \tau)\) be an IFTS and \( A = \langle x, \mu_A, \gamma_A \rangle \) be an IFS in \( X \). Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of \( A \) are defined by

\[
\text{int}(A) = \bigcup \{ G \mid G \text{ is an IFOS in } X \text{ and } G \subseteq A \},
\]

\[
\text{cl}(A) = \bigcap \{ K \mid K \text{ is an IFCS in } X \text{ and } A \subseteq K \}.
\]

Note that, for any IFS \( A \) in \((X, \tau)\), we have

\[
\text{cl}(\bar{A}) = \overline{\text{int}(A)} \quad \text{and} \quad \text{int}(\bar{A}) = \overline{\text{cl}(A)}.
\]

3. Intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi-precontinuous mappings

**Definition 3.1.** [5] An IFS \( A = \langle x, \mu_A, \gamma_A \rangle \) in an IFTS \((X, \tau)\) is said to be

(i) intuitionistic fuzzy semiopen if \( A \subseteq \text{cl}(\text{int}(A)) \),

(ii) intuitionistic fuzzy \( \alpha \)-open if \( A \subseteq \text{int}(\text{cl}(\text{int}(A))) \),

(iii) intuitionistic fuzzy preopen if \( A \subseteq \text{int}(\text{cl}(A)) \).

The family of all intuitionistic fuzzy semiopen (resp. intuitionistic fuzzy \( \alpha \)-open, intuitionistic fuzzy preopen) sets of an IFTS \((X, \tau)\) is denoted by \( \text{IFSO}(X) \) (resp. \( \text{IF}_{\alpha}(X), \text{IFPO}(X) \)). An IFS \( A = \langle x, \mu_A, \gamma_A \rangle \) in \((X, \tau)\) is said to be intuitionistic fuzzy semiclosed (resp. intuitionistic fuzzy \( \alpha \)-closed, intuitionistic fuzzy preclosed) if \( \bar{A} \in \text{IFSO}(X) \) (resp. \( \text{IF}_{\alpha}(X), \text{IFPO}(X) \)).

Every intuitionistic fuzzy open (resp. intuitionistic fuzzy closed) set is intuitionistic fuzzy \( \alpha \)-open (resp. intuitionistic fuzzy \( \alpha \)-closed), and every intuitionistic fuzzy \( \alpha \)-open (resp. intuitionistic fuzzy \( \alpha \)-closed) set is intuitionistic fuzzy semiopen (resp. intuitionistic fuzzy semiclosed) as well as intuitionistic fuzzy preopen (resp. intuitionistic fuzzy preclosed), but the separate converses need not be true in general (see [5]).

**Proposition 3.2.** Let \((X, \tau)\) be an IFTS and let \( A \in \text{IFS}(X) \). Then

\[ A \in \text{IFPO}(X) \iff (\exists B \in \tau) (A \subseteq B \subseteq \text{cl}(A)). \]
Definition 3.3. An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in an IFTS $(X, \tau)$ is said to be

(i) intuitionistic fuzzy semi-preopen if there exists $B \in IFSPO(X)$ such that $B \subseteq A \subseteq \text{cl}(B)$,

(ii) intuitionistic fuzzy semi-preclosed if there exists an intuitionistic fuzzy preclosed set $B$ such that $\text{int}(B) \subseteq A \subseteq B$.

The family of all intuitionistic fuzzy semi-preopen (resp. intuitionistic fuzzy semi-preclosed) sets of an IFTS $(X, \tau)$ will be denoted by $IFSPO(X)$ (resp. $IFSPC(X)$).

Example 3.4. Let $X = \{a, b, c\}$ and

$G_1 = \langle x, \left(\frac{a}{0.3}, \frac{b}{0.1}, \frac{c}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.4}\right) \rangle$,

$G_2 = \langle x, \left(\frac{a}{0.7}, \frac{b}{0.2}, \frac{c}{0.2}\right), \left(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.3}\right) \rangle$,

$G_3 = \langle x, \left(\frac{a}{0.7}, \frac{b}{0.2}, \frac{c}{0.2}\right), \left(\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}\right) \rangle$.

Then $\tau = \{0_-, 1_-, G_1, G_2\}$ is an IFT on $X$ and $G_3 \in IFSPO(X)$ (see [5, Example 2.13]). Let

$A = \langle x, \left(\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.2}\right), \left(\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{0.3}\right) \rangle$

be an IFS in $(X, \tau)$. Then $G_3 \subseteq A \subseteq \text{cl}(G_3)$, and hence $A \in IFSPO(X)$.

Theorem 3.5. For every IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in $(X, \tau)$, we have

$A \in IFSPC(X) \iff \overline{A} \in IFSPO(X)$.

Proof. Straightforward.

Every intuitionistic fuzzy semiopen (resp. intuitionistic fuzzy semiclosed) set and every intuitionistic fuzzy preopen (resp. intuitionistic fuzzy preclosed) set is intuitionistic fuzzy semi-preopen (resp. intuitionistic fuzzy semi-preclosed). But the separate converses need not be true in general.

Example 3.6. Let $X = \{a, b\}$ and

$A = \langle x, \left(\frac{a}{0.3}, \frac{b}{0.3}\right), \left(\frac{a}{0.3}, \frac{b}{0.3}\right) \rangle$,

$B = \langle x, \left(\frac{a}{0.7}, \frac{b}{0.2}\right), \left(\frac{a}{0.2}, \frac{b}{0.1}\right) \rangle$,

$C = \langle x, \left(\frac{a}{0.7}, \frac{b}{0.3}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}\right) \rangle$.

Then $\tau = \{0_-, 1_-, A\}$ is an IFT on $X$ and $B \in IFSPO(X)$. But $B \notin IFSO(X)$, because $B \not\subseteq A = \text{cl}(\text{int}(B))$. Also, $C \in IFSPO(X)$, but $C \notin IFPSO(X)$ since $C \not\subseteq 0_- = \text{int}(\text{cl}(C))$. 

Theorem 3.7. Let \((X, \tau)\) be an IFTS. Then

(i) Any union of intuitionistic fuzzy semi-preopen sets is intuitionistic fuzzy semi-preopen.

(ii) Any intersection of intuitionistic fuzzy semi-preclosed sets is intuitionistic fuzzy semi-preclosed.

Proof. (i) Let \(\{A_i\}_{i \in J}\) be a collection of intuitionistic fuzzy semi-preopen sets of \((X, \tau)\). Then there exists \(B_i \in \text{IFPO}(X)\) such that \(B_i \subseteq A_i \subseteq \text{cl}(B_i)\) for each \(i \in J\). It follows that \(\bigcup B_i \subseteq \bigcup A_i \subseteq \bigcup \text{cl}(B_i) \subseteq \text{cl}(\bigcup B_i)\) and \(\bigcup B_i \in \text{IFPO}(X)\).

(ii) is from (i) by taking complements. \(\square\)

The intersection of two intuitionistic fuzzy semi-preopen sets is not intuitionistic fuzzy semi-preopen in general as seen in the following example.

Example 3.8. Let \(X = \{a, b\}\) and
\[
A = \langle x, (0.3 \cdot 0.7), (0.6 \cdot 0.2) \rangle,
B = \langle x, (0.7 \cdot 0.2), (0.2 \cdot 0.5) \rangle.
\]
Then \(\tau = \{0, 1, \infty\}\) is an IFT on \(X, A \in \text{IFPO}(X)\) and \(B \in \text{IFSPO}(X)\). But \(A \cap B \notin \text{IFSPO}(X)\).

Theorem 3.9. For any IFS \(A = \langle x, \mu_A, \gamma_A \rangle\) in an IFTS \((X, \tau)\), \(A \in \text{IFSPO}(X)\) if and only if
\[
(\forall p(\alpha, \beta) \in A) (\exists B \in \text{IFSPO}(X)) (p(\alpha, \beta) \in B \subseteq A).
\]

Proof. If \(A \in \text{IFSPO}(X)\), then we can take \(B = A\) so that \(p(\alpha, \beta) \in B \subseteq A\) for every \(p(\alpha, \beta) \in A\). Let \(A\) be an IFS in \((X, \tau)\) and assume that there exists \(B \in \text{IFSPO}(X)\) such that \(p(\alpha, \beta) \in B \subseteq A\). Then
\[
A = \bigcup_{p(\alpha, \beta) \in A} \{p(\alpha, \beta)\} \subseteq \bigcup_{p(\alpha, \beta) \in A} B \subseteq A,
\]
and so \(A = \bigcup_{p(\alpha, \beta) \in A} B\) which is an intuitionistic fuzzy semi-preopen set by Theorem 3.7(i). \(\square\)

Theorem 3.10. Let \((X, \tau)\) be an IFTS. Then

(i) \((\forall A \in \text{IFSPO}(X)) (\forall B \in \text{IFS}(X)) (A \subseteq B \subseteq \text{cl}(A) \Rightarrow B \in \text{IFSPO}(X))\).

(ii) \((\forall A \in \text{IFSPC}(X)) (\forall B \in \text{IFS}(X)) (\text{int}(A) \subseteq B \subseteq A \Rightarrow B \in \text{IFSPC}(X))\).
Assume that \( A \subseteq B \subseteq \text{cl}(A) \) for every \( A \in \text{IFSPO}(X) \) and \( B \in \text{IFS}(X) \). Let \( C \in \text{IFPO}(X) \) be such that \( C \subseteq A \subseteq \text{cl}(C) \). Obviously, \( C \subseteq B \). From \( A \subseteq \text{cl}(C) \) it follows that \( \text{cl}(A) \subseteq \text{cl}(C) \) so that \( C \subseteq B \subseteq \text{cl}(A) \subseteq \text{cl}(C) \).

Hence \( B \in \text{IFSPO}(X) \).

(ii) follows from (i).

**Definition 3.11.** [5] Let \( f \) be a mapping from an IFTS \((X, \tau)\) into an IFTS \((Y, \kappa)\). Then \( f \) is said to be

(i) intuitionistic fuzzy semicontinuous if \( f^{-1}(B) \in \text{IFSO}(X) \) for every \( B \in \kappa \),

(ii) intuitionistic fuzzy \( \alpha \)-continuous if \( f^{-1}(B) \in \text{IF}_\alpha(X) \) for every \( B \in \kappa \),

(iii) intuitionistic fuzzy precontinuous if \( f^{-1}(B) \in \text{IFPO}(X) \) for every \( B \in \kappa \).

Every intuitionistic fuzzy continuous mapping is intuitionistic fuzzy \( \alpha \)-continuous and every intuitionistic fuzzy \( \alpha \)-continuous mapping is intuitionistic fuzzy semicontinuous as well as intuitionistic fuzzy precontinuous, but the separate converses may not be true in general (see [5]).

**Definition 3.12.** Let \((X, \tau)\) and \((Y, \kappa)\) be IFTSs. A mapping \( f : X \to Y \) is called an intuitionistic fuzzy semi-precontinuous if \( f^{-1}(B) \in \text{IFSPO}(X) \) for every \( B \in \kappa \).

Every intuitionistic fuzzy semicontinuous mapping is intuitionistic fuzzy semi-precontinuous but the converse may not be true as seen in the following example.

**Example 3.13.** Let \( X = \{a, b\}, Y = \{u, v\} \) and

\[
A = \langle x, \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{3}{5}} \right), \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{5}{2}} \right) \rangle, \\
B = \langle x, \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{3}{5}} \right), \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{5}{2}} \right) \rangle.
\]

Then \( \tau = \{0_\sim, 1_\sim, A\} \) and \( \kappa = \{0_\sim, 1_\sim, B\} \) are IFTs on \( X \) and \( Y \), respectively. Define a mapping \( f : (X, \tau) \to (Y, \kappa) \) by \( f(a) = u \) and \( f(b) = v \). Then \( f \) is intuitionistic fuzzy semi-precontinuous. Now we have

\[
f^{-1}(B) = \langle x, \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{3}{5}} \right), \left( \frac{a}{\frac{3}{5}}, \frac{b}{\frac{5}{2}} \right) \rangle,
\]

\[
\text{int}(f^{-1}(B)) = 0_\sim \cup A = A, \quad \text{and} \quad \text{cl}(\text{int}(f^{-1}(B))) = \text{cl}(A) = 1_\sim \cap A = A.
\]

Thus \( f^{-1}(B) \not\subseteq \text{cl}(\text{int}(f^{-1}(B))) \), which shows that \( f \) is not intuitionistic fuzzy semi-precontinuous.

Every intuitionistic fuzzy precontinuous mapping is intuitionistic fuzzy semi-precontinuous but the converse may not be true as seen in the following example.
Example 3.14. Let $X = \{a, b\}$, $Y = \{u, v\}$ and

$$A = \langle x, \left(\frac{a}{0.3}, \frac{b}{0.3}\right), \left(\frac{a}{0.7}, \frac{b}{0.6}\right)\rangle,$$

$$B = \langle x, \left(\frac{a}{0.5}, \frac{b}{0.5}\right)\rangle.$$

Then $\tau = \{0_\sim, 1_\sim, A\}$ and $\kappa = \{0_\sim, 1_\sim, B\}$ are IFTs on $X$ and $Y$, respectively. Define a mapping $g : (X, \tau) \to (Y, \kappa)$ by $g(a) = u$ and $g(b) = v$. Then $g$ is intuitionistic fuzzy semi-precontinuous but not intuitionistic fuzzy precontinuous. In fact, we have

$$g^{-1}(B) = \langle x, \left(\frac{a}{0.6}, \frac{b}{0.5}\right), \left(\frac{a}{0.4}, \frac{b}{0.5}\right)\rangle,$$

$$\text{cl}(g^{-1}(B)) = 1_\sim \cap \bar{A} = \bar{A}$$

and

$$\text{int}(\text{cl}(g^{-1}(B))) = \text{int}(\bar{A}) = 0_\sim \cup A = A.$$

Hence $g^{-1}(B) \nsubseteq A = \text{int}(\text{cl}(g^{-1}(B)))$, and so $g$ is not intuitionistic fuzzy precontinuous.

Theorem 3.15. Let $f : X \to Y$ be a mapping from an IFTS $X$ to an IFTS $Y$. Then the following assertions are equivalent:

(i) $f$ is intuitionistic fuzzy semi-precontinuous.

(ii) $f^{-1}(B) \in \text{IFSPC}(X)$ for every IFCS $B$ in $Y$.

(iii) for every intuitionistic fuzzy point $p_{(\alpha, \beta)}$ in $X$ and every IFOS $B$ in $Y$ such that $f(p_{(\alpha, \beta)}) \in B$, there exists $A \in \text{IFSPO}(X)$ such that $p_{(\alpha, \beta)} \in A$ and $f(A) \subseteq B$.

Proof. (i) $\Rightarrow$ (ii) Obvious.

(i) $\Rightarrow$ (iii) Assume that $f$ is intuitionistic fuzzy semi-precontinuous. Let $p_{(\alpha, \beta)}$ be an intuitionistic fuzzy point in $X$ and $B$ be an IFOS in $Y$ such that $f(p_{(\alpha, \beta)}) \in B$. Take $A = f^{-1}(B)$. Then $A \in \text{IFSPO}(X)$ by the definition of intuitionistic fuzzy semi-precontinuous, and

$$p_{(\alpha, \beta)} \in f^{-1}(f(p_{(\alpha, \beta)})) = f^{-1}(B) = A$$

and $f(A) = f(f^{-1}(B)) \subseteq B$.

(iii) $\Rightarrow$ (i) Let $B$ be an IFOS in $Y$ and $p_{(\alpha, \beta)} \in f^{-1}(B)$. Then $f(p_{(\alpha, \beta)}) \in B$. Using (iii), we know that there exists $A \in \text{IFSPO}(X)$ such that $p_{(\alpha, \beta)} \in A$ and $f(A) \subseteq B$. It follows that $p_{(\alpha, \beta)} \in A \subseteq f^{-1}(B)$ so from Theorem 3.9 that $f^{-1}(B) \in \text{IFSPO}(X)$. Thus $f$ is intuitionistic fuzzy semi-precontinuous. □
References


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