NILPOTENCY INDEX OF NIL-ALGEBRA OF NIL-INDEX 3

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ABSTRACT. Nagata and Higman proved that any nil-algebra of finite nil-index is nilpotent of finite index. The Nagata-Higman Theorem can be formulated in terms of T-ideals. The T-ideal generated by $a^n$ for all $a \in A$ is also generated by the symmetric polynomials. The symmetric polynomials play an important role in analyzing nil-algebra. We construct the incidence matrix with the symmetric polynomials. Using this incidence matrix, we determine the nilpotency index of nil-algebra of nil-index 3.

AMS Mathematics Subject Classification : 15A72
Key words and phrases : Nil-algebra, nilpotency index, nil-index.

1. Introduction.

Let $F$ be a field of characteristic 0 and $A$ be a $F$-algebra. We do not assume that $A$ has units. Suppose that there is a positive integer $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$, then $A$ is called a nil-algebra and the natural number $n$ is called the nil-index of $A$. If $A^m = 0$, but $A^{m-1} \neq 0$, then $A$ is said to be nilpotent of index $m$ or $A$ has nilpotency $m$. In other words, $A$ is nilpotent of index $m$ if any product of $m$ elements of $A$ is 0.

Theorem 1.1. [1, 4] (Nagata-Higman Theorem) If there exists a positive integer $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in A$, then there is a natural number $N(n)$ such that $a_1 a_2 \cdots a_N = 0$ for all $a_1, a_2, \ldots, a_N \in A$.

Let’s review the brief history of this theorem and the following consequence. Nagata in 1953 first proved the above theorem. Higman provided a simpler proof in 1956 and showed that

$$\frac{n^2}{e^2} \leq N(n) \leq 2^n - 1.$$

Received June 3, 2005.
It was Razmyslov [6] in 1974 who improved Higman’s upper bound to $n^2$, that is, $N(n) \leq n^2$. It was Kuzmin [2] in 1975 who heightened the lower bound to $n^{(n+1)/2}$, i.e.,

$$\frac{n(n + 1)}{2} \leq N(n).$$

It is known that $N(2) = 3$. In this work, we show that $N(3) = 6$. Determining $N(n)$ for $n \geq 4$ is still an open question[3].

Let $C(n)$ be the commutative ring (with unit) generated by the traces, $T(\sigma)$ where $\sigma$ is a monomial in the generic $n \times n$ matrices $X_1, X_2, \ldots$ and $R(n)$ be the ring generated by $C(n)$ and the generic $n \times n$ matrices. Let’s denote by $R^+(n)$ the two-sided ideal of $R(n)$ generated by $X_1, X_2, \ldots$. Let $P(n)$ be the least integer such that $R^+(n)$ is generated as a $C(n)$-module by monomials in the generic matrices $X_1, X_2, \ldots$ of degree $\leq P(n)$. And $Q(n)$ is the least integer such that $C(n)$ is generated as a $F$-algebra by elements of degree $\leq Q(n)$.

**Theorem 1.2.** [5] The natural numbers $N(n)$, $P(n) + 1$ and $Q(n)$ are equal.

**2. Symmetric Polynomials and Incidence matrices.**

Let’s begin with the definition of symmetric polynomial.

**Definition 2.1.** The symmetric polynomial of degree $n$ is the polynomial

$$S_n(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in \text{Sym}(n)} a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)},$$

where $\text{Sym}(n)$ is the symmetric group on $n$ letters.

Let $T_n$ be the $T$-ideal generated by $a^n$ for $a \in A$. For example, when $n = 2$, $a_1^2$, $a_2^2 \in T_2$, also $(a_1 + a_2)^2 \in T_2$. Thus $a_1^2 + a_1a_2 + a_2a_1 + a_2^2 \in T_2$ implies

$$S_2(a_1, a_2) = a_1a_2 + a_2a_1 \in T_2.$$

By multilinearizing, we can show that $T_n$ is generated by the symmetric polynomial $S_n(a_1, a_2, \ldots, a_n)$ of degree $n$ for $a_1, a_2, \ldots, a_n \in A$.

Given a partition $P = (p_1, \ldots, p_m)$ of $m$ with $n$ parts, where $p_i \leq p_{i+1}$, $1 \leq i \leq n - 1$, the incidence matrix denoted by $(n, m)^P$ is constructed as the following. We label the columns by the (multilinear) monomials of $m$ variables lexicographically. In other words, the first column is labeled by $a_1 a_2 \cdots a_{m-2} a_{m-1} a_m$, the second by $a_1 a_2 \cdots a_{m-2} a_m a_{m-1}$ and so on. Thus the last column is indexed by $a_m a_{m-1} a_{m-2} \cdots a_2 a_1$. We use $1, 2, \ldots, a_1, a_2, \ldots$, if there is no risk of confusion. Suppose that $j$-th column is indexed by
Table 1. The \((2,3)^{(1,2)}\)-incidence matrix.

<table>
<thead>
<tr>
<th></th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_2(1,23))</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(S_2(1,32))</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(S_2(2,13))</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(S_2(2,31))</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(S_2(3,12))</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(S_2(3,21))</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\(i_1 \cdots i_m\). Then \(j\)-th row of the incidence matrix corresponding to the partition \(P = (p_1, \ldots, p_n)\) is labeled by

\[S_n(i_1 \cdots i_{p_1}, i_{p_1+1} \cdots i_{p_1+p_2}, \ldots, i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n}).\]

In the \(j\)-th row, one places 1 for the columns labeled by the monomials that appear in that row index, and 0 elsewhere. In other words, if

\[S_n(i_1 \cdots i_{p_1}, i_{p_1+1} \cdots i_{p_1+p_2}, \ldots, i_{p_1+\cdots+p_{n-1}+1} \cdots i_{p_1+\cdots+p_n}) = i_1 \cdots i_{p_1} i_{p_1+1} \cdots i_{p_1+p_2} \cdots i_{p_1+p_2+p_3} \cdots + \cdots + \]

then put 1 for the columns labeled by

\[i_1 \cdots i_{p_1} i_{p_1+1} \cdots i_{p_1+p_2} \cdots i_{p_1+p_2+p_3} \cdots \]

For instance, the \((2,3)^{(1,2)}\)-incidence matrix is in Table 1. This matrix has determinant 4 and this fact implies that the nil-algebra of nil-index 2 is nilpotent of index 3.

### 3. Properties of Incidence matrices.

Let \((1^{r_1} 2^{r_2} \cdots k^{r_k})\) be a partition of \(m\) with \(n\) parts where the superscript \(r_i\) is the multiplicity of part of size \(i\). Hence \(1 \cdot r_1 + 2 \cdot r_2 + \cdots + k \cdot r_k = m\) and \(r_1 + r_2 + \cdots + r_k = n\).

**Proposition 3.1.** The \((n,m)^P\)-incidence matrix is a \([0,1]\)-matrix, with constant row and column sums \(n!\) for both.
Proof. The $\langle n, m \rangle^P$-incidence matrix has $m!$ columns. In each row, we assign 1 for all the monomials of
\[ S_n(i_1, \ldots, i_{p_1}, i_{p_1+1}, \ldots, i_{p_1+p_2}, \ldots, i_{p_1+\ldots+p_{n-1}+1}, \ldots, i_{p_1+\ldots+p_n}), \]
which contains $n!$ monomials. Therefore the row sum of $\langle n, m \rangle^P$ is $n!$. By the symmetry of $\langle n, m \rangle^P$, we only need to determine the sum of the first column, which is labeled by $12\cdots m = a_1a_2\cdots a_m$. The number of ways dividing the string $12\cdots m$ into $n$ sets is
\[ \frac{n!}{\prod_{i=1}^{k} r_i!}. \]
But in each division, one gets $\prod_{i=1}^{k} r_i!$ distinct monomials. So the column sum of $\langle n, m \rangle^P$ is
\[ \prod_{i=1}^{k} r_i! \cdot \frac{n!}{\prod_{i=1}^{k} r_i!} = n!. \]
Even though we have the column sum of $\langle n, m \rangle^P$ as $n!$, some of the columns can be identical. The number of the distinct rows in the $\langle n, m \rangle^P$ is
\[ \frac{n!}{\prod_{i=1}^{k} r_i!}. \]

Now we are ready to prove our main theorem.

**Theorem 3.2.** The nil-algebra $A$ of nil-index 3 is nilpotent of index 6.

Proof. The partition of 6 with 3 parts are $(1, 1, 4), (1, 2, 3)$ and $(2, 2, 2)$. By the Proposition 3.1, the $(3, 6)$-incidence matrix is of dimension $6! = 720$. Each of $(3, 6)^{(1,1,4)}, (3, 6)^{(1,2,3)}$ and $(3, 6)^{(2,2,2)}$ has the row and column sums $3! = 6$. Furthermore $(3, 6)^{(1,1,4)}$ has $\frac{6!}{2!} = 360$ distinct rows and $(3, 6)^{(2,2,2)}$ has $\frac{6!}{3!} = 120$ distinct rows. This means that $(3, 6)^{(1,1,4)}$ and $(3, 6)^{(2,2,2)}$ have ranks less than 720, which implies none of them are invertible. If we consider the sum of $(3, 6)^{(1,1,4)}$, $(3, 6)^{(1,2,3)}$ and $(3, 6)^{(2,2,2)}$, the resulting matrix has nonzero determinant
\[ 2^{208}3^{291}5^{10}7^{16}11^{10}13^{10}17^{14}23^{9}29^{6}67^{5} \]
which implies that the sum of 3 incidence matrices is invertible. Thus the monomial $a_1a_2\cdots a_6$ can be expressed by the sum of $S_3(w_1, w_2, w_3)$ where $w_i$ are words in $a_1, \ldots, a_6$ and $\sum_{i=1}^{3} \text{length}(w_i) = 6$. \qed
References


Woo Lee received his BA from Sogang Univ and Ph.D at Pennsylvania State Univ. under the supervision of Edward Formanek. Since 1998, he has been at Kwangju University. His research interests include noncommutative rings, braid groups and representations.
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