COHOMOLOGY GROUPS OF RADICAL EXTENSIONS

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Abstract. If $k$ is a subfield of $\mathbb{Q}(\varepsilon_m)$ then the cohomology group $H^2(k(\varepsilon_n)/k)$ is isomorphic to $H^2(k(\varepsilon_{n'})/k)$ with $\gcd(m, n') = 1$. This enables us to reduce a cyclotomic $k$-algebra over $k(\varepsilon_n)$ to the one over $k(\varepsilon_{n'})$. A radical extension in projective Schur algebra theory is regarded as an analog of cyclotomic extension in Schur algebra theory. We will study a reduction of cohomology group of radical extension and show that a Galois cohomology group of a radical extension is isomorphic to that of a certain subextension of radical extension. We then draw a cohomological characterization of radical group.

1. Introduction

Let $k$ be a field, $k^*$ be the multiplicative subgroup of $k$ and $\mu(k)$ be the group of roots of unity in $k$. For a Galois extension $L$ of $k$ with Galois group $G = \text{Gal}(L/k)$ and for a 2-cocycle $\alpha \in Z^2(G, L^*) = Z^2(L/k, L^*)$, a crossed product algebra $(L/k, \alpha) = \sum_{\sigma \in G} L u_\sigma$ with $u_\sigma u_\tau = \alpha(\sigma, \tau) u_{\sigma\tau}$ and $u_\sigma x = \sigma(x) u_\sigma$ ($x \in L$, $\sigma, \tau \in G$) is called a cyclotomic algebra if $L$ is a cyclotomic extension of $k$ and $\alpha$ has values in $\mu(L)$ (i.e., $\alpha \in Z^2(L/k, \mu(L))$). Let $H^2(\mu(L)/k)$ be the image of a canonical homomorphism $\iota$ of $H^2(\mu(L)/\mu(L))$ into $H^2(L/k, L^*)$ induced by the inclusion $\mu(L) \hookrightarrow L^*$. Since $\mu(L)$ is a subgroup of the torsion group of $L^*$, $\iota$ is injective ([7, p.91]), so we may identify $H^2(\mu(L)/k) = H^2(\mu(L)/L^*)$. Suppose $k$ is a subfield of the cyclotomic extension $\mathbb{Q}(\varepsilon_m)$ ($\mathbb{Q}$: the rational number field, $\varepsilon_m$: a primitive $m$-th root of unity). Let $L = k(\varepsilon_n)$. Due to [6], $m$ and $n$ are assumed to be either odd or divisible by $4$. Then the Galois cohomology group $H^2(L/k)$ is isomorphic to $H^2(K/k)$ where $K = k(\varepsilon_{n'})$ is a subextension of $L$ such that $n'$ is a certain divisor of $n$ which is prime to $m$ ([13, (7.12)]. Employing this result, Janusz's reduction theorem on cyclotomic algebras in [6] ([13, (7.9)]) follows that, a cyclotomic algebra $(L/k, \alpha)$ with $\alpha \in Z^2(L/k, \mu(L))$ can be reduced to the case $\gcd(m, n) = 1$, i.e., to $(K/k, \beta)$ where $\beta$ is a 2-cocycle in $Z^2(K/k, \mu(K))$ defined over the smaller group $G(K/k)$.
It is well known that every Schur k-algebra (a central simple k-algebra which
is a homomorphic image of a group algebra kG for a finite group G) is similar
to a cyclotomic k-algebra [13, (3.10)]. The idea of Schur algebra has been
generalized to a projective Schur algebra in [9] by replacing group algebra by
twisted group algebra; a projective Schur k-algebra is a central simple algebra
that is a homomorphic image of a twisted group algebra k G for a finite group
G and α ∈ Z2(G, k*).

The analogue of cyclotomic algebra in the theory of projective Schur algebra
is the radical algebra ([1]). A radical k-algebra is a crossed product algebra
(L/k, α) where L = k(Ω) is a finite Galois radical extension of k, Ω is a subgroup
of L* which is finite modulo k* (i.e., Ω/k* finite), and α ∈ Z2(L/k, L*) is
represented by a 2-cocycle with values in Ω.

In this paper we study radical extensions and radical algebras, and obtain a
respective result to Janusz’s reduction theorem on radical extensions. We
derive a reduction of Galois cohomology groups over radical extension fields,
indeed prove that for a radical extension L of k, there exists a Galois radical
extension K of k in L such that the cohomology group of G(L/k) is iso-
morphic to that of G(K/k) (in Theorem 10). We then verify a cohomologi-
ical characterization of radical groups that a homomorphism of radical groups
R(K/k) → R(L/k) commutes with certain homomorphisms of cohomology
groups (in Theorem 14).

All notations are standard. H2(L/k, M) is the 2-dimensional cohomology
group H2(G, M) where G = G(L/k) and M is a G-module, while Z2(L/k, M)
is the 2-cocycle group. If M = L*, we write H2(L/k, L*) = H2(L/k). Let εd
(d > 0) denote a primitive d-th root of unity, a|b denote the division of b by a,
while a''|b denote the highest power t of a to be a''|b.

2. Preliminaries

Lemma 1. ([13, 7.10]) Let H be a cyclic normal subgroup of G and M be a fi-
nite G-module. Let Nh = h∈H h. If Nh(M) = M then inf : H2(G/H, M/H) → H2(G, M) is an isomorphism, where inf is the inflation map from G/H to
G and M/H is the subset of M consisting of elements fixed by H.

For finite Galois extensions K and L of k with K < L, the norm N_{L/K}:
L → K, x → \prod_{\sigma \in G(L/K)} \sigma(x) is a homomorphism for x ∈ L. If H = G(L/K)
is normal in G = G(L/k), then N_{L/K} corresponds to Nh in Lemma 1. In particular it is clear that N_{Q(\varepsilon_{p^i})/Q(\varepsilon_{p^i+1})} \varepsilon_{p^i+1} = \varepsilon_{p^i+1}^H for a prime
p and i > 0, thus the following theorem is due to Lemma 1.

Theorem 2. ([13, (7.12)]) Let k ≤ Q(\varepsilon_m) and L = Q(\varepsilon_m, \varepsilon_n). Let n' = 4^\delta p_1 \cdots p_s where p_i are distinct odd prime divisors of n not dividing m, and
δ = 1 if 4 | n, 4 | m; δ = 0 otherwise. Let K = Q(\varepsilon_m, \varepsilon_{n'}). Then H2(K/k) ≅ H2(L/k).

As a consequence of Theorem 2, Janusz proved the next theorem on algebras.
Theorem 3. ([6], [13, (7.9)]) If \( k \leq \mathbb{Q}(\varepsilon_m) \) then any cyclotomic algebra over \( k \) is similar to the cyclotomic algebra \( (\mathbb{Q}(\varepsilon_m, \varepsilon_1)/k, \alpha) \) with \( t = 4^s p_1 \cdots p_s; \delta = 0 \) if \( 4|m \) and \( \delta = 1 \) otherwise, where all \( p_i \) are distinct odd primes not dividing \( m \).

Theorem 2 and 3 can be generalized to any cyclotomic extension field \( L \) containing finitely many roots of unity in \([4]\).

Theorem 4. Let \( k \leq \mathbb{Q}(\varepsilon_m) \) and \( L = \mathbb{Q}(\varepsilon_{m_1}, \varepsilon_{m_2}, \ldots, \varepsilon_{m_w}) \). Let

\[
n'_1 = 4^{s_1}p_1 \cdots p_s \quad \text{with distinct odd primes} \quad p_j|n_1, \; p_j \nmid m \quad (1 \leq j \leq s)
\]

and \( \delta = 1 \) if \( 4|n_1, \; 4 \nmid m; \delta = 0 \) otherwise. And for \( 1 < i \leq w \), let

\[
n'_i = 4^{s_i}p_1 \cdots p_{s_i}
\]

with \( p_{ij}|n_1, \; p_j \nmid n_v(1 \leq j < i) \) where \( p_{ij} \) are distinct odd primes, and \( \delta = 1 \) if \( 4|n_1, \; 4 \nmid m, \; 4 \nmid n_v \; (1 \leq v < i); \delta = 0 \) otherwise.

Then \( H_2^2(L/k) \cong H_2^2(K/k) \) where \( K = \mathbb{Q}(\varepsilon_{m_1}, \varepsilon_{n'_1}, \ldots, \varepsilon_{n'_w}) \). Furthermore, a crossed product algebra \((L/k, \alpha)\) with \( \alpha \in Z^2(L/k, \mu(L)) \) is similar to \((K/k, \beta)\) where \( \beta \) is a 2-cocycle having values in \( \mu(K) \).

Proof. We will prove this when \( L = \mathbb{Q}(\varepsilon_m, \varepsilon_{m_1}, \varepsilon_{m_2}) \). Write \( n'_1 = 4^{s_1}p_1 \cdots p_s \) and \( n'_2 = 4^{s_2}q_1 \cdots q_s \) where \( p_i \) [resp. \( q_i \)] are distinct odd prime divisors of \( n_1 \) [resp. \( n_2 \)] with \( p_i \nmid m, \; q_j \nmid m \) and \( q_j \nmid n_1 \) for \( 1 \leq i \leq s, \; 1 \leq j \leq u \). And \( \delta = 1 \) if \( 4|n_1, \; 4 \nmid m; \delta = 2 \) if \( 4|n_2, \; 4 \nmid m, \; 4 \nmid n_1; \) and \( \delta = 0 \) otherwise.

Let \( K = \mathbb{Q}(\varepsilon_m, \varepsilon_{n'_1}, \varepsilon_{n'_2}) \) and \( E = \mathbb{Q}(\varepsilon_m, \varepsilon_{n'_1}, \varepsilon_{n'_2}) \). Then

\[
k < K = \mathbb{Q}(\varepsilon_{mn'_1}, \varepsilon_{mn'_2}) \leq E = \mathbb{Q}(\varepsilon_{mn'_1}, \varepsilon_{n'_2})
\]

for, \( \gcd(m, n'_i) = 1 \) \((i = 1, 2) \). Since each \( q_j \) in the factorization of \( n'_2 \) satisfies

\[
q_j \nmid mn'_1; \quad \delta = 1 \quad \text{if} \quad 4|n_2 \quad \text{and} \quad 4 \nmid mn'_1; \quad \text{and} \quad \delta = 0 \quad \text{otherwise},
\]

we are able to use Theorem 2 on \( K \) and \( E \) to get \( H_2^2(K/k) \cong H_2^2(E/k) \).

Moreover with \( l = \text{lcm}(m, n_2) \), we have

\[
E = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_2})(\varepsilon_{n'_2}) = \mathbb{Q}(\varepsilon_1, \varepsilon_{n'_1}) \leq L = \mathbb{Q}(\varepsilon_m, \varepsilon_{n_2})(\varepsilon_{n_1}) = \mathbb{Q}(\varepsilon_1, \varepsilon_{n_1})
\]

and each \( p_i \) in the factorization of \( n'_1 \) satisfies

\[
p_i \nmid l; \quad \delta = 1 \quad \text{if} \quad 4|n_1 \quad \text{and} \quad 4 \nmid l; \quad \text{and} \quad \delta = 0 \quad \text{otherwise}.
\]

Thus by applying Theorem 2 to \( L \) and \( E \), we have \( H_2^2(E/k) \cong H_2^2(L/k) \).

Now the isomorphism \((L/k, \alpha) \cong (K/k, \beta)\) of crossed product algebra with \( \alpha \in Z^2(L/k, \mu(L)), \beta \in Z^2(K/k, \mu(K)) \) follows immediately by Theorem 3. \( \square \)
3. Norm on radical extension fields

A field $L$ is a (finite) radical extension of $k$ if there is a subgroup $\Omega$ of $L^*$ such that $L = k(\Omega)$ and $\Omega/k^*$ is a (finite) torsion group. We may exhibit $L$ by $k(\sqrt[N]{a_1}, \ldots, \sqrt[N]{a_w})$ with $a_i \in k^*$ and $n_i > 0$ ($1 \leq i \leq w$). Moreover if $L = k(\Omega)$ is a Galois extension over $k$, then $\Omega$ is $G(L/k)$-invariant, so $k(\Omega)$ contains enough roots of unity, i.e., $L = k(\{\sqrt[n_i]{\epsilon_{n_i}} | 1 \leq i \leq w\})$ with primitive $n_i$-th roots of unity $\epsilon_{n_i}$. Clearly $\Omega$ is not determined uniquely, while $\Omega/k^*$ is unique and finite.

The most interesting case of a cyclic extension is $L = k(\lambda)$ with $\lambda \in L^*$ and $\lambda^n \in k$. If $L/k$ is Galois radical then we may regard $L = k(\epsilon_n, \lambda)$. In particular if $\epsilon_n \in k$ then $L/k$ is a cyclic extension of degree dividing $n$(see [8, Theorem 14.4]). The radical extension $L = k(\lambda)$ is said to be irreducible if degree $[L : k] = n$. Thus $L/k$ is irreducible radical if and only if $L = k(\lambda)$ and $\lambda \in L^*$ is a root of an irreducible polynomial $X^n - a \in k[X]$, and, if and only if the order of $\lambda k^*$ in $L^*/k^*$ is equal to the degree of $\lambda$ over $k$.

Remark that, in case of a cyclotomic extension $L = \mathbb{Q}(\sqrt[n]{m}, \sqrt[n]{n_1}, \ldots, \sqrt[n]{n_w})$, the reduction in Theorem 4 would follow immediately by taking $n = \text{lcm}(n_1, \ldots, n_w)$ and applying Theorem 2 to $m$ and $n$. However when $L = k(\sqrt[n_1]{m}, \ldots, \sqrt[n_w]{m})$ is a Galois radical extension, to get a kind of reduction we need to know how to choose $n_i$ from $n_i$ each other explicitly, not just taking $n_i$ from $n = \text{lcm}$ of $n_i$'s.

**Lemma 5.** Let $L = k(\Omega)$ be a finite Galois radical extension of $k$. Then $\Omega/k^*$ is a finite $G$-module where $G = \mathcal{G}(L/k)$.

**Proof.** The finite group $\Omega/k^*$ has a $G$-module structure by defining an action $\sigma \lambda = \sigma(\lambda') k^*$ where $\sigma \in G$ and $\lambda \in \Omega/k^*$ such that $\lambda = \lambda' k^*$ for $\lambda' \in \Omega$. \hfill \Box

We begin with a radical extension containing an $n$-th root of $1 \neq a \in k^*$.

**Theorem 6.** Let $k = \mathbb{Q}(\sqrt[n]{m})$ and $K = k(\sqrt[n]{a})$ be a Galois radical extension of $k$, where $m$ and $n$ are positive odd integers. For a prime $p$ dividing $n$, let $E = k(\sqrt[p]{\sqrt[n]{m}})$ and $F = k(\sqrt[p]{\sqrt[n]{a}})$ be Galois radical extensions of $k$. Then

(i) $E = F$, or $E/F$ is cyclic of degree $p$ and $N_{E/F}(\sqrt[p]{\sqrt[n]{m}}) = \langle \sqrt[p]{\sqrt[n]{m}} \rangle$.

(ii) Furthermore we assume $p^i|m$ and $p^b|n$.

a) If $0 \leq z \leq b$ then $F/K$ is cyclic of degree $p$, and $N_{F/K}(\sqrt[p]{\sqrt[n]{m}}, \sqrt[p]{\sqrt[n]{a}}) = \langle \sqrt[p]{\sqrt[n]{m}} \rangle$.

b) If $b < z$ then $E = K$, or $F = K$ so $N_{E/K}(\sqrt[p]{\sqrt[n]{m}}, \sqrt[p]{\sqrt[n]{a}}) = \langle \sqrt[p]{\sqrt[n]{m}} \rangle$.

**Proof.** The Galois radical extensions $K$ and $E$ form

$$k < K = k(\sqrt[n]{m}, \sqrt[n]{a}) < F = k(\sqrt[p]{\sqrt[n]{m}}, \sqrt[p]{\sqrt[n]{a}}) < E = k(\sqrt[p]{\sqrt[n]{m}}, \sqrt[p]{\sqrt[n]{a}}).$$

Since $E = k(\sqrt[p]{\sqrt[n]{a}}) = F(\sqrt[p]{\lambda})$ where $\lambda = \sqrt[n]{a} \in F$ and $\sqrt[p]{\lambda}$ is a root of $X^p - \sqrt[n]{a} \in F[X]$, and since $\sqrt[p]{\sqrt[n]{m}} \in F$, $E/F$ is a cyclic extension of degree dividing $p$ [8, Theorem 14.4]. Hence $E = F$ or $[E : F] = p$. 


If \([E : F] = p\), any \(\sigma \in \mathcal{G}(E/F)\) maps \(\sqrt[n]{a}\) to a zero of \(X^n - a\), say \(\sigma(\sqrt[n]{a}) = e_{pn}^{\lambda} \sqrt[n]{a}\) for \(0 \leq \lambda < pn\). But \(\sqrt[n]{a^n}\) is a zero of \(X^n - a\), so \(\sqrt[n]{a^n} \in F^*\), and \(\sigma(\sqrt[n]{a^n}) = e_{pn}^{\lambda^n} \sqrt[n]{a^n}\) and \(e_{pn}^{\lambda^n} = 1\). Thus \(l = tn\) for some \(t = 0, \ldots, p-1\).

Moreover since each \(\sigma \in \mathcal{G}(E/F)\) leaves \(\varepsilon_{pn}\) fixed, we have
\[
\sigma(\varepsilon_{pn}^{m,n} \sqrt[n]{a} \varepsilon_{pn}^2(\sqrt[n]{a}) = \varepsilon_{pn}^{2m,n} \sqrt[n]{a}, \ldots, \sigma^{p-1}(\sqrt[n]{a}) = \varepsilon_{pn}^{(p-1)tn} \sqrt[n]{a},
\]
hence it follows that
\[
N_{E/F}(\sqrt[n]{a}) = \prod_{\sigma \in \mathcal{G}(E/F)} \sigma(\sqrt[n]{a}) = e_{pn}^{tn(1+\cdots+(p-1))} \varepsilon_{pn}^{\lambda^n} = \varepsilon_{pn}^{\lambda^n} \sqrt[n]{a^n}.
\]
Thus \(N_{E/F}(\sqrt[n]{a}) = \langle \varepsilon_{pn}^{\lambda^n} \rangle = \langle \sqrt[n]{a^n} \rangle\) and \(N_{E/F}(\varepsilon_{pn}) = \langle \varepsilon_{pn}^{p^n} \rangle = \langle \varepsilon_n \rangle\).

Suppose that \(m = p^nm'\) and \(n = p^nn'\) with \(p \nmid m'n'\). If \(0 \leq z \leq b\) then
\[
lcm(m, pn) = \frac{p^nm'p^{b+1}n'}{p^z\gcd(m', p^{b+1}z^n')} = \frac{m'p^{b+1}n'}{\gcd(m', n')} = p^{b+1}lcm(m', n')
\]
and similarly \(lcm(m, n) = p^z\gcd(m', n')\). Set \(lcm(m', n') = l\). Then \(Q(\varepsilon_m, \varepsilon_{pn}) = Q(\varepsilon_{pn+1})\) and \(Q(\varepsilon_m, \varepsilon_n) = Q(\varepsilon_{p^n})\) because \(p \nmid l\), hence
\[
[Q(\varepsilon_m, \varepsilon_{pn}) : Q(\varepsilon_m, \varepsilon_n)] = \frac{\phi(p^{b+1})}{\phi(p^n)} = \phi(p^{b+1}) = p
\]
(\(\phi\) is the Euler phi function), which shows that \([F : K] = p\).

Now each \(\tau \in \mathcal{G}(F/K)\) maps \(\varepsilon_{pn} \in F\) to \(e_{pn}^{\lambda} \varepsilon_{pn}\) for some \(0 \leq \lambda < pn\). Because \(e_{pn}^{h_p} \in K^*\), we have \(e_{pn}^{h_p} = \tau(e_{pn}^{h_p}) = e_{pn}^{h_p}\), hence \(p(l - 1) \equiv 0 \pmod{pn}\) and \(l = tn + 1\) with \(0 \leq t < p\). Indeed, the cyclic group \(\mathcal{G}(F/K)\) of order \(p\) consists of automorphisms \(\tau_t\) such that \(\tau_t(e_{pn}) = e_{pn}^{tn+1}\) for \(0 < t < p\). Thus we have
\[
N_{F/K}(\varepsilon_{pn}) = \sum_{t=0}^{t=p-1} e_{pn}^{tn+1} = e_{pn}^{p+pn} = e_{pn}^p,
\]
so \(N_{F/K}(\varepsilon_{pn}) = \langle e_{pn}^p \rangle \leq \langle \varepsilon_n \rangle\). Comparing the orders, we get \(N_{F/K}(\varepsilon_{pn}) = \langle \varepsilon_n \rangle\).

And \(N_{F/K}(\sqrt[n]{a}) = \langle \prod_{\tau \in \mathcal{G}(F/K)} \tau(\sqrt[n]{a}) \rangle = \langle \sqrt[n]{a^n} \rangle < \langle \varepsilon_n, \sqrt[n]{a} \rangle\), this is (ii-a).

In case of \(m = p^nm', n = p^nn'\) with \(0 < b < z\), \(gcd(m, pn) = pgcd(m, n)\), \(lcm(m, pn) = lcm(m, n)\), so \(Q(\varepsilon_m, \varepsilon_{pn}) = Q(\varepsilon_m, \varepsilon_n) = Q(\varepsilon_{pn})\) and \([F : K] = 1\). Hence \(E = K\), or \([E : K] = p\) and \(N_{E/K}(\varepsilon_{pn}, \sqrt[n]{a}) = N_{E/F}(\varepsilon_{pn}, \sqrt[n]{a}) = \langle \varepsilon_n, \sqrt[n]{a} \rangle\).

We observe that the assumption in (ii-b), i.e., \(m = p^nm', n = p^nn'\) with \(b < z\) implies that \(k = Q(\varepsilon_m)\) already contains enough \(p\)-th roots of unity.

**Corollary 7.** Assume the same context as in (ii-b) Theorem 6 for radical extensions \(K = k(\sqrt[n]{a}) = k(\Omega_K)\) and \(E = k(\sqrt[n]{a}) = k(\Omega_E)\) with finite \(\Omega_K/k^*\) and \(\Omega_E/k^*\). If \(H = \mathcal{G}(E/K) \neq 1\) then \(N_H(\Omega_E/k^*) = \Omega_K/k^*\).

**Proof.** \(E = k(\varepsilon_{pn}, \sqrt[n]{a})\) is a cyclic extension of \(K = k(\varepsilon_n, \sqrt[n]{a})\) of degree \(p\) with cyclic Galois group \(H\). The mapping \(N_H = \prod_{\sigma \in H} \sigma\) in Lemma 1 determines \(N_H(\varepsilon_{pn}, \sqrt[n]{a}) = \langle \varepsilon_n, \sqrt[n]{a} \rangle\) by Theorem 6. By the action in Lemma 5, we have
\[ N_H(\Omega_E/k^*) = N_H(\varepsilon_n k^*, \sqrt[n]{ak^*}) = \langle \varepsilon_n k^*, \sqrt[n]{ak^*} \rangle = \Omega_K/k^*. \]

We shall denote \( N_H(\Omega_E/k^*) \) by the same notation \( N_{E/k}(\Omega_E/k^*) \) which is the norm map. As a generalization of Theorem 6, we have the following theorem.

**Theorem 8.** Let \( k = \mathbb{Q}(\varepsilon_m) \) and \( L = k(\sqrt[n]{a}) = k(\Omega) \) be a Galois radical extension of \( k \). Assume a prime \( p \) with \( p^i \not| m \) and \( p^k \not| n \). Let \( F = k(\sqrt[n]{a}) = k(\Omega_F) \) be a Galois radical extension with \( L \neq F \). If \( 0 < b \leq z \) then \( N_{L/F}(\Omega/k^*) = \Omega_{F/k^*}. \)

**Proof.** Let \( E = k(\varepsilon_n, \sqrt[n]{a}) \) be a Galois radical extension. Then

\[ k < F = k(\varepsilon_{n/p}, \sqrt[n/p]{a}) < E = k(\varepsilon_n, \sqrt[n]{a}) < L = k(\varepsilon_n, \sqrt[n]{a}), \]

and \( L = E(\sqrt[n]{\lambda}) \) where \( \lambda = \sqrt[n]{\lambda} \in E \) and \( \sqrt[n]{\lambda} \) is a root of a polynomial \( X^p - \sqrt[n]{\lambda} \) in \( E[X] \). Since \( \varepsilon_n \in E \), \( L/E \) is a cyclic radical extension of degree dividing \( p \). So \( L = E \) or \( [L : E] = p \). Because \( b \leq z \), \( k \) contains enough \( p \)-th roots of unity so \( F = E \). But since \( L \neq F \), we have \( [L : F] = p \) and

\[ N_{L/F}(\varepsilon_n, \sqrt[n]{a}) = \langle \varepsilon_{n/p}, \sqrt[n/p]{a} \rangle = \langle \varepsilon_n, \sqrt[n]{a} \rangle. \]

Similar to Corollary 7, we consequently have \( N_{L/F}(\Omega/k^*) = \Omega_{F/k^*}. \)

### 4. Cohomology group on radical extension fields

We shall discuss cohomology groups over Galois radical extension fields, and have a reduction of cohomology group that is an analog of Theorem 2. Let \( H \) be a normal subgroup of \( G \) and \( M \) be a \( G \)-module. The inflation-restriction sequence on cohomology group

\[
1 \to H^r(G/H, M^H) \xrightarrow{\text{inf}} H^r(G, M) \xrightarrow{\text{res}} H^r(H, M)
\]

is exact if \( r = 1 \). When \( r > 1 \), the sequence is exact if \( H^i(H, M) = 1 \) for all \( 1 \leq i \leq r - 1 \) (refer to [12, (3.4.2), (3.2.3)]).

**Theorem 9.** Let \( n = p^b n_0 \) and \( m = p^k n_0 \) with an odd prime \( p \not| n_0m_0 \). Let \( k = \mathbb{Q}(\varepsilon_m) \), and \( L = k(\sqrt[n]{a}) = k(\Omega) \) and \( L_0 = k(\sqrt[n]{a}) = k(\Omega_0) \) be Galois radical extensions of \( k \) with \( L \neq L_0 \). If \( 0 < b \leq z \) then the inflation map is an isomorphism on cohomology groups

\[ H^2(L_0/k, \Omega_0/k^*) \xrightarrow{\text{inf}} H^2(L/k, \Omega_0/k^*). \]

**Proof.** Obviously \( L = k(\varepsilon_n, \sqrt[n]{a}) \) and \( L_0 = k(\varepsilon_{n_0}, \sqrt[n_0]{a}). \) Since \( b \leq z, \varepsilon_{p^c} \) is contained in \( L_0 \) so \( \varepsilon_n \in L_0 \). Thus \( L = L_0(\sqrt[n]{\lambda}) \) where \( \lambda = \sqrt[n]{\lambda} \) and \( \sqrt[n]{\lambda} \) is a root of \( X^p - \sqrt[n]{\lambda} \) in \( L_0[X] \). So \( L/L_0 \) is cyclic of degree \( p^c \) for some \( c \leq b \).

Due to Lemma 5, \( \Omega/k^* \) is a \( G(L/k) \)-module with module action that, for any \( \tau \in G(L/k) \) and \( \sqrt[n]{\lambda} k^* \in \Omega/k^* \), \( \tau(\sqrt[n]{\lambda} k^*) = \tau(\sqrt[n]{\lambda} k^*) = \varepsilon_n^{\frac{c}{p^c}} \sqrt[n]{\lambda} k^* \in \Omega/k^* \)
for some $l > 0$. Similarly $\Omega_0/k^*$ is a $G(L_0/k)$-module. Moreover it is easy to see that $\Omega_0/k^*$ is also a $G(L/k)$-module by regarding $\sqrt[n]{a}$ as $(\sqrt[n]{a})^{p^b}$.

Write $H = G(L/L_0)$. From the sequence of groups

$$1 \to G(L/L_0) \to G(L/k) \to G(L_0/k) \to 1,$$

we consider the inflation-restriction sequence

$$H^2(L_0/k, (\Omega_0/k^*)^H) \xrightarrow{\inf} H^2(L/k, \Omega_0/k^*) \xrightarrow{\text{res}} H^2(L/L_0, \Omega_0/k^*).$$

Since $H$ is cyclic, it follows from [12, (1.5.6)] and [12, (3.2.1)] that

$$H^2(L/L_0, \Omega_0/k^*) = H^0(H_0, \Omega_0/k^*) = \frac{(\Omega_0/k^*)^H}{N_{L/L_0}(\Omega_0/k^*)}.$$

Every $\sigma \in H$ leaves all elements in $\Omega_0/k^*$ fixed, so $(\Omega_0/k^*)^H = \Omega_0/k^*$. Moreover due to Theorem 8, we compute the norm $N_{L/L_0}$ directly to get

$$N_{L/L_0}(\Omega_0/k^*) = \langle \epsilon_z^{p^b} \rangle, \quad \langle \sqrt[n]{a}^{p^b} \rangle = \langle \epsilon_z^{p^b} \rangle = \Omega_0/k^*$$

for $\gcd(p^i, n) = 1$. Hence $H^2(L/L_0, \Omega_0/k^*) = 1$.

Again since $H$ is finite cyclic and $\Omega_0/k^*$ is a finite $H$-module, we use the Herbrand’s quotient of $\Omega_0/k^*$ (refer to [3, (23.2)]) that

$$1 = h_2/1(\Omega_0/k^*) = \frac{|H^2(L/L_0, \Omega_0/k^*)|}{|H^1(L/L_0, \Omega_0/k^*)|}$$

hence it follows that $H^1(L/L_0, \Omega_0/k^*) = H^2(L/L_0, \Omega_0/k^*) = 1$. Therefore we can conclude from (1) that the sequence

$$1 \to H^2(L_0/k, (\Omega_0/k^*)^H) \to H^2(L/k, \Omega_0/k^*) \to H^2(L/L_0, \Omega_0/k^*) = 1$$

is exact, so have an isomorphism $H^2(L_0/k, \Omega_0/k^*) \cong H^2(L/k, \Omega_0/k^*)$.

For a given finite radical extension $L$ over $k$, we observed in Theorem 9 that the cohomology group over $G(L/k)$ can be decreased down to that over $G(L_0/k)$, where $L_0$ is the Galois radical extension of $k$ smaller than $L$ by ‘one’ prime factor power $p^b$. We can strengthen this observation with the following theorem which will go to reduction of Galois cohomology groups.

**Theorem 10.** Let $k = \mathbb{Q}(\sqrt[n]{a})$ and $L = k(\sqrt[n]{a})$ be the same fields as in Theorem 9. Assume $n = p_1^{b_1} \cdots p_u^{b_u}$ with distinct primes $p_i$ and $b_i > 0$. For each $p_i$, $(1 \leq i \leq u)$, write $m = p_i^{z_i} m_i$ (with $p_i \mid m_i$ and $z_i \geq 0$). Suppose $b_i \leq z_i$ for some $1 \leq i \leq u$, and after appropriate renumbering, we assume that $b_i \leq z_i$ for all $s + 1 \leq i \leq u$. Let $n_0 = p_1^{b_1} \cdots p_s^{b_s}$, and let the Galois extensions be

$$L_0 = k(\sqrt[n]{a}) = k(\Omega_0),$$

$$L_j = k(\sqrt[n]{a^{p_{s+1}^{b_{s+1}} \cdots p_{s+j}^{b_{s+j}}}}) = k(\Omega_j) \quad \text{for } 1 \leq j \leq u - s.$$ 

Then $H^2(L_j/k, \Omega_{j-1}/k^*) \cong H^2(L_{j-1}/k, \Omega_{j-1}/k^*)$ for all $1 \leq j \leq u - s$, so $H^2(L/k, \Omega_0/k^*)$ and $H^2(L_0/k, \Omega_0/k^*)$ are isomorphic.
Proof. Since $L_0$, $L_j$ ($1 \leq j \leq u - s$) are all Galois radical extensions of $k$, and since $n = n_1 \cdots n_{b_s} \cdots n_{b_{s+1}} \cdots n_{b_u} = n_0 \cdot n_{b_{s+1}} \cdots n_{b_u}$, we have $L_{u-s} = L$ and

$$k < L_0 = k(\varepsilon_{n_0}, \sqrt[n]{a}) < L_1 = k(\varepsilon_{n_{b_{s+1}}}, \sqrt[n_{b_{s+1}}]{a}) < \cdots < L_j = k(\varepsilon_{n_{b_{s+1}} \cdots b_{s+j}}, \sqrt[n_{b_{s+1}} \cdots b_{s+j}]{a}) < \cdots < L_{u-s} = L.$$  

Since $m = p_1^* m_i$ and $0 < b_i < z_i$ for $s + 1 \leq i \leq u$, each $\varepsilon_{p_i} \in k$. Together with $\varepsilon_{n_0} \in L_0$, $\varepsilon_{n_{b_{s+1}} \cdots b_{s+j}}$ belong to $L_0$ for all $1 \leq j \leq u - s$. Hence $L_j = L_{j-1}(\lambda_j)$, where $\lambda_j \in L_j$ and $\lambda_{j+1}^{p_{s+j}} \in L_{j-1}$ ($1 \leq j \leq u - s$), and $L_j/L_{j-1}$ is cyclic of degree $p_{s+j}^c$ for some $c_j \leq b_{s+j}$. Thus we have the isomorphism on cohomology groups

$$H^2(L_1/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*)$$

due to Theorem 9, and proceeding inductively we get

(2) $H^2(L_j/k, \Omega_{j-1}/k^*) \cong H^2(L_{j-1}/k, \Omega_{j-1}/k^*)$ for each $1 \leq j \leq u - s$.

Let $H = G(L_2/L_0)$, and consider the inflation-restriction sequence

(3) $H^2(L_2/k, \Omega_0/k^*) \xrightarrow{\text{inf}} H^2(L_2/k, \Omega_0/k^*) \xrightarrow{\text{res}} H^2(L_2/L_0, \Omega_0/k^*)$.

Since $L_2 = L_0(\xi)$ with $\xi^{p_{s+1} \cdot p_{s+2}} \in L_0$, $L_2/L_0$ is a cyclic radical extension of degree dividing $p_{s+1} \cdot p_{s+2}$, and in fact $[L_2 : L_0] = p_{s+1}^c p_{s+2}^c$. Thus $H = G(L_2/L_0) \cong \mathbb{Z}^{p_{s+1}^c} \times \mathbb{Z}^{p_{s+2}^c} \cong G(L_2/L_1) \times G(L_1/L_0)$, and by [7, (2.3.14)], we have

$$H^2(L_2/L_0, \Omega_0/k^*) \cong H^2(L_2/L_1, \Omega_0/k^*) \times H^2(L_1/L_0, \Omega_0/k^*).$$

But since $p_i \nmid n_0$ for $s + 1 \leq i \leq u$, we obtain

$$N_{L_1/L_0}(\Omega_0/k^*) = (\varepsilon_{n_0}, \sqrt[n_0]{k^*})^{p_{s+1}} = \Omega_0/k^* = (\Omega_0/k^*)^{L_1/L_0}$$

and similarly, $N_{L_2/L_1}(\Omega_0/k^*) = (\Omega_0/k^*)^{p_{s+2}} = \Omega_0/k^* = (\Omega_0/k^*)^{L_2/L_1}$.

By the proof of Theorem 9, both $H^2(L_1/L_0, \Omega_0/k^*)$ and $H^2(L_2/L_1, \Omega_0/k^*)$ are trivial groups, so that $H^2(L_2/L_0, \Omega_0/k^*) = 1$. Then again the Herbrand quotient of $\Omega_0/k^*$ is equal to 1, i.e.,

$$1 = |H^2(L_2/L_0, \Omega_0/k^*)|/|H^1(L_2/L_0, \Omega_0/k^*)|.$$  

So $H^1(L_2/L_0, \Omega_0/k^*)$ is trivial, and the inflation-restriction sequence (3) is exact. We thus obtain the isomorphism on cohomology groups

$$H^2(L_0/k, \Omega_0/k^*) \cong H^2(L_2/k, \Omega_0/k^*).$$

And together with (2), it follows that

$$H^2(L_2/k, \Omega_0/k^*) \cong H^2(L_1/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*).$$
Applying this process to the cyclic group \( H = G(L_3/L_0) \) of order \( p_a^c p_b^c \) and to the sequence

\[
H^2(L_0/k, (\Omega_0/k^*))^H \xrightarrow{\text{inf}} H^2(L_3/k, \Omega_0/k^*) \xrightarrow{\text{res}} H^2(L_3/L_0, \Omega_0/k^*),
\]
we also get \( H^2(L_3/L_0, \Omega_0/k^*) = 1 \), for \( G(L_3/L_0) \cong G(L_3/L_2) \times G(L_2/L_0) \) and \( H^2(L_3/L_2, \Omega_0/k^*) = 1 = H^2(L_2/L_0, \Omega_0/k^*) \).

The exactness of the inflation-restriction sequence guarantees the isomorphism \( H^2(L_3/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*) \). Continuing, we eventually get

\[
H^2(L/k, \Omega_0/k^*) = H^2(L_{u+1}/k, \Omega_0/k^*) \cong H^2(L_0/k, \Omega_0/k^*)
\]

\( \square \)

Remark 1. In Theorem 10, we assume \( a = 1 \), i.e., \( \sqrt{\alpha} = \varepsilon_n \) for \( n = p_1^b \cdots p_u^b \) (odd primes). Suppose \( m = p_i^z m_i \) with \( z_i < b_i \) for \( 1 \leq i \leq s \), and \( b_i < z_i \) for \( s + 1 \leq i \leq u \). Let \( n_0 = p_1^b \cdots p_u^b \), \( L = \mathbb{Q}(\varepsilon_n, \varepsilon_{n_0}) \) and \( L_0 = \mathbb{Q}(\varepsilon_n) \).

Then \( L = L_0 \) and \( H^2(L/k, \Omega_L/k^*) \cong H^2(L_0/k, \Omega_{L_0}/k^*) \) (this is Theorem 10). Moreover, by rearrangement if necessary, we assume \( z_1 = \cdots = z_t = 0 \) for \( t \leq s \), i.e., \( p_1 \mid m \) for \( 1 \leq i \leq t \). By letting \( n_0' = p_1 \cdots p_t \), we can further reduce the cohomology group by Janusz theorem that

\[
H^2(L/k, \mu(L)) = H^2(L_0/k, \mu(L_0)) \cong H^2(k(\varepsilon_{n_0}^c)/k, \mu(k(\varepsilon_{n_0}^c))).
\]

Remark 2. It is natural to ask whether the Remark 1 is true for \( a \neq 1 \) with the same \( n \) and \( n_0' \), i.e., is \( H^2(k(\sqrt{\alpha})/k, (\sqrt{\alpha})/k^*) \cong H^2(k(\sqrt{\alpha})/k, (\sqrt{\alpha})/k^*) \)? The following proposition provides a partial answer.

Proposition 11. Let \( L = k(\sqrt{\alpha}) \) be a Galois radical extension of \( k = \mathbb{Q}(\varepsilon_n) \).

Let \( n = p_1^b \cdots p_u^b \) (\( b_i > 0 \), odd primes \( p_i \)) and \( m = p_i^z m_i \) (\( z_i \geq 0 \), \( p_i \nmid m_i \)) for all \( i \). By rearrangement, assume \( z_i < b_i \) for \( 1 \leq i \leq s (\leq u) \), \( z_i > b_i \) for \( s + 1 < i \leq u \), and moreover \( z_j = 0 \) for \( 1 \leq j \leq t (\leq s) \). Set \( n_0 = p_1^b \cdots p_u^b \) and \( n_0' = p_1 \cdots p_t \).

Let \( K = k(\sqrt[3]{\alpha}) = k(\Omega) \), and \( F_v = k(\varepsilon_{n_0}/p_v^{b_v-2}, k, \sqrt[3]{\alpha}) \) be Galois radical extensions of \( k \) for \( t + 1 \leq v \leq s \).

Then \( H^2(L/k, \Omega/k^*) \cong H^2(F_v/k, \Omega/k^*) \), \( H^2(K/k, \Omega/k^*) \cong H^2(F_v/k, \Omega/k^*) \), but \( H^2(F_v/k, \Omega/k^*) \neq H^2(B_v/k, \Omega/k^*) \).

Proof. Let \( L_0 = k(\sqrt[3]{\alpha}) = k(\Omega_{L_0}) \) be a Galois radical extension of \( k \). With the integers \( n = p_1^b \cdots p_u^b, n_0 = p_1^b \cdots p_u^b \) and \( n_0' = p_1 \cdots p_t \) for \( t \leq s \leq u \), it is clear that \( k < K < L_0 < L \) and \( H^2(L/k, \Omega_{L_0}/k^*) \cong H^2(L_0/k, \Omega_{L_0}/k^*) \) due to Theorem 10. Hence it follows that

\[
H^2(L/k, \Omega/k^*) \cong H^2(L_0/k, \Omega/k^*)
\]

Let \( p_v \) be one of \( p_{t+1}, \ldots, p_s \). Since \( m = p_v^{c_v} m_v \) with \( 0 < c_v \leq b_v \), the Galois radical extensions of \( k \) are

\[
K = k(\varepsilon_{n_0'}, \sqrt[3]{\alpha}) < F_v = k(\varepsilon_{n_0}/p_v^{b_v-2}, \sqrt[3]{\alpha}) < L_0 = k(\varepsilon_{n_0}, \sqrt[3]{\alpha}).
\]
Now let $\mathcal{K}$, $\mathcal{F}_v$ and $\mathcal{L}_0$ denote the cyclotomic extensions

$$
\mathcal{K} = \mathbb{Q}(\varepsilon_{n_0}, \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}}) < \mathcal{F}_v = \mathbb{Q}(\varepsilon_{m_0}, \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}}) < \mathcal{L}_0 = \mathbb{Q}(\varepsilon_{m_0}, \varepsilon_{n_0}).
$$

Owing to $p_{v^*}^* || \frac{m_0}{p_{v^*}}$ and $\varepsilon_{p_{v^*}}^* \in \mathcal{K}$, we have $p_v \nmid [\mathcal{F}_v : \mathcal{K}]$ and $H^2(\mathcal{F}_v/\mathcal{K}, \mu(\mathcal{F}_v)) = H^2(\mathcal{L}_0/\mathcal{F}_v, \mu(\mathcal{L}_0)) = 1$. Thus due to Janusz theorem we have

$$
H^2(\mathcal{K}/k, \mu(\mathcal{K})) \cong H^2(\mathcal{F}_v/k, \mu(\mathcal{F}_v)) \cong H^2(\mathcal{L}_0/k, \mu(\mathcal{L}_0)).
$$

On the other hand, from the tower of fields

$$
k < K = \mathcal{K}(\sqrt[n]{\alpha}) < F_v = \mathcal{F}_v(\varepsilon_{m_0}, \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}}) < \mathcal{L}_0 = \mathcal{L}_0(\sqrt[n]{\alpha}),
$$

consider a field $A_v = k(\varepsilon_{n_1}, \varepsilon_{n_0}^{\frac{1}{p^*}})$. We now observe the followings.

(i) $K < B_v < F_v < A_v < \mathcal{L}_0$

(ii) $L_0 = A_v(\sqrt[n]{\lambda})$ where $\lambda = \varepsilon_{n_0}^{\frac{1}{p^*}}\sqrt[n]{\alpha}$ and $\sqrt[n]{\lambda}$ is a root of $X^{p^*} - \lambda \in A_v[X]$. Since $\varepsilon_{p^*}^* \in A_v$, $L_0/A_v$ is a cyclic extension of order $p^*_v$ with $w_v \leq z_v$.

(iii) Similar to (ii), it can be seen that $F_v = B_v(\varepsilon_{n_0}^{\frac{1}{p^*}}\sqrt[n]{\beta})$ where $\beta = \varepsilon_{n_0}^{\frac{1}{p^*}}\sqrt[n]{\alpha} \in B_v$ and $\varepsilon_{n_0}^{\frac{1}{p^*}}\sqrt[n]{\beta} \in B_v[X]$. Since $\varepsilon_{n_0}^{\frac{1}{p^*}}\varepsilon_{n_0}^{\frac{2}{p^*}}\varepsilon_{n_0}^{\frac{3}{p^*}}$ belongs to $B_v = \mathbb{Q}(\varepsilon_{m_0}, \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}})$, $F_v/B_v$ is cyclic of degree $v_{l^*}/v_{l^*} = v_{l^*}/v_{l^*}$ dividing $p_{l^*} - 1, \ldots, p_{l^*} - 1, \ldots, p_{l^*} - 1$.

(iv) $A_v = F_v(\varepsilon_{n_0}^{\frac{1}{p^*}})$, so $G(A_v/F_v) \cong G(\mathcal{L}_0/F_v)$.

(v) $B_v = K(\varepsilon_{n_0}^{\frac{1}{p^*}})$, so $G(B_v/K) \cong G(F_v/K)$.

Thus from (ii), $L_0/A_v$ is cyclic of order $p^*_v$ with $w_v \leq z_v$, so

$$
N_{L_0/A_v}(\Omega/k^*) = N_{L_0/A_v}(\varepsilon_{m_0}, \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}}) \cong \langle \varepsilon_{n_0}^{\frac{1}{p^*}}, \varepsilon_{n_0}^{\frac{2}{p^*}}, \varepsilon_{n_0}^{\frac{3}{p^*}} \rangle,
$$

and this is equal to $\Omega/k^*$ because $\gcd(n_0, p_v) = 1$ for $t + 1 \leq v \leq s$. Hence

$$
H^2(L_0/A_v, \Omega/k^*) = \frac{(\Omega/k^*)^2}{N_{L_0/A_v}(\Omega/k^*)} = 1
$$

$$
= H^2(L_0/A_v, \Omega/k^*) = H^2(L_0/A_v, \Omega/k^*)
$$

So the exact sequence

$$
H^2(A_v/k, \Omega/k^*) \to H^2(L_0/k, \Omega/k^*) \to H^2(L_0/A_v, \Omega/k^*)
$$

yields the isomorphism $H^2(A_v/k, \Omega/k^*) \cong H^2(L_0/k, \Omega/k^*)$. 

From (iv), \( A_v = F_v(\varepsilon_{p^n m - v}) \) and \( \mathcal{G}(A_v/F_v) \cong \mathcal{G}(L_0/F_v) \) cyclic, so the invariant set \( \Omega/k^* \) by \( \mathcal{G}(A_v/F_v) \) corresponds to \( \mu(K)/k^* \) by \( \mathcal{G}(L_0/F_v) \). Thus
\[
H^2(A_v/F_v, \Omega/k^*) \cong H^2(L_0/F_v, \mu(K)/k^*).
\]
Since \( H^2(L_0/F_v, \mu(K)) \cong H^2(L_0/F_v, \mu(L_0)) = 1 \) by Janusz theorem, we have
\[
H^2(L_0/F_v, \mu(K)) = 1.
\]
Moreover since \( H^2(L_0/F_v, \mu(K)) \to H^2(L_0/F_v, \mu(K)/k^*) \), we have
\[
H^2(L_0/F_v, \mu(K)/k^*) = 1 = H^2(A_v/F_v, \Omega/k^*).
\]
So we obtain the isomorphism \( H^2(F_v/k, \Omega/k^*) \cong H^2(A/k, \Omega/k^*) \) from the exact sequence \( H^2(F_v/k, \Omega/k^*) \to H^2(A_v/k, \Omega/k^*) \to H^2(A_v/F_v, \Omega/k^*) = 1 \).

We therefore have the isomorphisms
\[
H^2(L/k, \Omega/k^*) \cong H^2(L_0/k, \Omega/k^*) \cong H^2(A_v/k, \Omega/k^*) \cong H^2(F_v/k, \Omega/k^*).
\]

Now from (v), \( B_v = K(\varepsilon_{n_0/p^n m - v}) \) and \( \mathcal{G}(B_v/K) \cong \mathcal{G}(F_v/K) \) cyclic. As above,
\[
H^2(B_v/K, \Omega/k^*) \cong H^2(F_v/K, \mu(K)/k^*) = 1.
\]
Thus \( H^2(K/k, \Omega/k^*) \to H^2(B_v/k, \Omega/k^*) \to H^2(B_v/K, \Omega/k^*) = 1 \) is exact, so the isomorphism \( H^2(K/k, \Omega/k^*) \cong H^2(B_v/k, \Omega/k^*) \) follows.

However we observe that \( H^2(B_v/k, \Omega/k^*) \) is not isomorphic to \( H^2(F_v/k, \Omega/k^*) \). In fact, \( F_v/B_v \) is cyclic of degree \( d \) dividing \( p_{i-1}^{b_i} \cdots p_i^{b_{i+1}} \cdots p_v^{b_v} \cdots p_{w}^{b_w} \) by (iii). Thus the \( N_{F_v/B_v}(\Omega/k^*) = N_{F_v/B_v}(\varepsilon m k^*, \varepsilon n_0^* k^*, \varepsilon_{m} k^*, \varepsilon_{m}^d k^*, (\varepsilon_{m}^d k^*) \neq \Omega/k^* \), because \( \gcd(d, n_0) \) need not be 1.

The exact correspondence of Theorem 2 with respect to radical extension is to show \( H^2(K/k, \Omega_K/k^*) \cong H^2(L/k, \Omega_L/k^*) \) where \( K = k(\Omega_K) < L = k(\Omega_L) \). Instead of this, we proved in Theorem 10 that \( H^2(K/k, \Omega_K/k^*) \cong H^2(L/k, \Omega_L/k^*) \) which is a subgroup of \( H^2(L/k, \Omega_L/k^*) \). We have discussed a radical extension field with one \( n \)-th root of an element in \( k \). The next theorem is about a radical extension having more than one \( n \)-th root.

**Theorem 12.** Let \( k = \mathbb{Q}(\varepsilon_m) \). Write \( m = p^m m' \) and \( n_i = p^i n'_i (i = 1, 2) \) with an odd prime \( p \) \( \nmid m n'_1 n'_2 \), and \( z, b_i \geq 0 \). Let \( L = k(\sqrt[2]{a_1}, \sqrt[2]{a_2}) = k(\Omega_L) \), \( F = k(\sqrt[2]{a_1}, \sqrt[2]{a_2}) = k(\Omega_F) \), and \( K = k(\sqrt[2]{a_1}, \sqrt[2]{a_2}) = k(\Omega) \) be Galois radical extensions of \( k \). Assume \( b_i \leq z \) for \( i = 1, 2 \). Then
(i) \( N_{F/K}(\Omega_F/k^*) = \Omega/k^* \cong N_{L/K}(\Omega_L/k^*) \).
(ii) Moreover, \( H^2(L/k, \Omega/k^*) \cong H^2(F/k, \Omega/k^*) \cong H^2(K/k, \Omega/k^*) \).

**Proof.** We may write the Galois radical extensions of \( k \) by
\[
K = k(\varepsilon_{n_1}, \sqrt[2]{a_1}, \sqrt[2]{a_2}) = F = k(\varepsilon_{n_1}, \varepsilon_{n_2}, \sqrt[2]{a_1}, \sqrt[2]{a_2}) \,< L = k(\varepsilon_{n_1}, \sqrt[2]{a_1}, \sqrt[2]{a_2})
\]
(i = 1, 2). Since \( p^{b_i} \leq p^2 \) and \( \varepsilon_{p_i} \in k \), we have \( \varepsilon_{p_i} \in k < K \). Together with \( \varepsilon_{n_i} \in K \), it follows that \( \varepsilon_{n_i} \) belongs to \( K \). Hence

\[ F = k(\sqrt[p]{\psi_1}, \sqrt[p]{\psi_2}) = K(\sqrt[p^2]{\lambda_2}), \quad \text{where} \quad \lambda_2 = \sqrt[p]{\psi_2} \in K \]

and \( \sqrt[p^2]{\lambda_2} \) is a root of \( X^{p^2} - \lambda_2 \in K[\lambda] \). Thus \( F/K \) is a cyclic extension of degree \( p^2 \) with \( c_2 = \lambda_2 \). And the minimal polynomial over \( K \) of \( \sqrt[p^2]{\lambda_2} \in F \) is \( X^{p^2} - \sqrt[p^2]{\lambda_2} \in K[\lambda] \). Thus \( \sqrt[p^2]{\psi_2} \in K \) and \( \sqrt[p^2]{\omega_2} \in \langle \psi_2 \rangle \).

Moreover the cyclic group \( G(F/K) \) is generated by \( \sigma \) such that

\[ \sigma(\sqrt[p^2]{\lambda_2}) = \varepsilon_{p^2}^{\sqrt[p^2]{\lambda_2}}, \quad \text{i.e.,} \quad \sigma(\sqrt[p^2]{\psi_2}) = \varepsilon_{n_2}^{\sqrt[p^2]{\lambda_2}} \varepsilon_{\omega_2}. \]

Now for the Galois extension \( L \) over \( F \),

\[ L = k(\sqrt[p]{\psi_1}, \sqrt[p]{\psi_2}) = F(\sqrt[p^2]{\lambda_1}), \quad \text{where} \quad \lambda_1 = \sqrt[p]{\psi_1} \in F \]

and \( \sqrt[p^2]{\lambda_1} \) is a root of \( X^{p^2} - \lambda_1 \in F[\lambda] \). Since \( \varepsilon_{p^2} \in F \), \( L/F \) is cyclic of degree \( p^2 \) for \( c_1 = \lambda_1 \). Then \( \sqrt[p^2]{\psi_1} \in F \) and

\[ G(L/F) = \langle \tau \rangle \quad \text{such that} \quad \tau(\sqrt[p^2]{\psi_1}) = \varepsilon_{n_1}^{\sqrt[p^2]{\lambda_1}} \varepsilon_{\omega_1}. \]

We shall compute the norm \( N_{F/K} \) on \( \Omega_F/k^* = \langle \varepsilon_{n_1}, \varepsilon_{n_2}, \sqrt[p^2]{\psi}, \sqrt[p^2]{\omega} \rangle \) that

\[ N_{F/K}(\varepsilon_{n_1}k^*) = \left( \prod_{i=0}^{p^2-1} \sigma^i(\varepsilon_{n_1}k^*) \right) = \langle \varepsilon_{n_1}^{p^2}k^* \rangle \leq \langle \varepsilon_{n_1}k^* \rangle, \]

and the equality holds because \( 1 = \gcd(n_1, p) \). Similarly

\[ N_{F/K}(\varepsilon_{n_2}k^*) = N_{F/K}(\varepsilon_{n_2}^{p^2}k^*) = \langle \varepsilon_{n_2}k^* \rangle, \]

\[ N_{F/K}(\sqrt[p^2]{\psi}k^*) = \left( \prod_{i=0}^{p^2-1} \sigma^i(\sqrt[p^2]{\psi}k^*) \right) = \langle \sqrt[p^2]{\psi}^{p^2}k^* \rangle \leq \langle \sqrt[p^2]{\psi}k^* \rangle, \]

and the equality \( N_{F/K}(\sqrt[p^2]{\omega}k^*) = \langle \sqrt[p^2]{\omega}k^* \rangle \) holds, for \( 1 = \gcd(n_1, p) \). Moreover

\[ N_{F/K}(\sqrt[p^2]{\psi_2}k^*) = \left( \prod_{i=0}^{p^2-1} \sigma^i(\sqrt[p^2]{\psi_2}k^*) \right) \]

\[ = \langle \varepsilon_{n_2}^{p^2-2}n_2(1 + \cdots + (p^2-1)) \sqrt[p^2]{\omega_2}^{p^2}k^* \rangle \]

\[ \leq \langle \sqrt[p^2]{\omega_2}^{p^2}k^* \rangle \leq K, \quad \text{i.e.,} \quad \langle \sqrt[p^2]{\omega_2}^{p^2}k^* \rangle \leq \langle \sqrt[p^2]{\omega_2}k^* \rangle \leq K. \]

Since orders of \( \sqrt[p^2]{\psi_2} \) and \( \sqrt[p^2]{\omega_2} \) over \( K \) are \( n_2p^{b_2-c_2} \) and \( n_2' \), respectively, and \( n_2'p^{b_2-c_2} \geq n_2 \), the equality \( N_{F/K}(\sqrt[p^2]{\omega_2}k^*) = \langle \sqrt[p^2]{\omega_2}k^* \rangle \) follows. Thus we have

\[ N_{F/K}(\Omega_F/k^*) = \langle \varepsilon_{n_1}k^*, \varepsilon_{n_2}k^*, \sqrt[p^2]{\psi_2}k^*, \sqrt[p^2]{\omega_2}k^* \rangle = \Omega/k^*. \]
On the other hand, we shall observe that $N_{L/F}(\Omega_{L/k}) \neq \Omega_{F/k^*}$. In fact, since $G(L/F) = \langle \tau \rangle$ with $\tau(\sqrt[p^{i-1}]{a}) = \varepsilon_{n_i}^{(p^{i-1})n_i} \sqrt[p^{i-1}]{a}$ for $0 \leq i \leq p^c - 1$, and since $\varepsilon_{n_1}, \varepsilon_{n_2} \in K < F$, it is easy to see that

\[ N_{L/F}(\varepsilon_{n_1} k^*) = \prod_i \tau(\varepsilon_{n_1} k^*) = \prod_i \varepsilon_{n_1}^{(p^{i-1})n_i} k^* \]

and

\[ N_{L/F}(\varepsilon_{n_2} k^*) = \prod_i \tau(\varepsilon_{n_2} k^*) = \prod_i \varepsilon_{n_2}^{(p^{i-1})n_i} k^* \]

and

\[ N_{L/F}(\sqrt[p]{a} k^*) = \prod_i \tau(\sqrt[p]{a} k^*) = \prod_i \varepsilon_{n_1}^{(p^{i-1})n_i} \sqrt[p]{a} k^* \]

(Products run over $0 \leq i \leq p^c - 1$). Comparing the orders $|\langle \sqrt[p^{i-1}]{a} k^* \rangle| = n_i^{p^{i-1} \geq n_i} = \langle \sqrt[p]{a} k^* \rangle$ over $F$, we have $N_{L/F}(\sqrt[p]{a} k^*) = \langle \sqrt[p]{a} k^* \rangle$. But

\[ N_{L/F}(\sqrt[p]{a} k^*) = \prod_{i=0}^{p^c-1} \tau(\sqrt[p]{a} k^*) = \langle \sqrt[p]{a} k^* \rangle \]

However, we will show that $N_{L/K}(\Omega_{L/k}) = \Omega_{K/k^*}$. Owing to the chain rule of the norm map, we obtain

$N_{L/K}(\varepsilon_{n_1} k^*) = N_{F/K}(\varepsilon_{n_1} k^*) = \langle \varepsilon_{n_1} k^* \rangle$, and similarly

$N_{L/K}(\varepsilon_{n_2} k^*) = \langle \varepsilon_{n_2} k^* \rangle$ and $N_{L/K}(\sqrt[p]{a} k^*) = \langle \sqrt[p]{a} k^* \rangle$. Furthermore

\[ N_{L/K}(\sqrt[p]{a} k^*) = N_{F/K}(\sqrt[p]{a} k^*) = \prod_{i=0}^{p^c-1} \sigma(\sqrt[p]{a} k^*) \]

\[ = \langle \varepsilon_{n_1}^{(p^{i-1})n_i} \sqrt[p]{a} k^* \rangle = \langle \sqrt[p]{a} k^* \rangle. \]

Hence it follows that

$N_{L/K}(\Omega_{L/k}) = \langle \varepsilon_{n_1} k^*, \varepsilon_{n_2} k^*, \sqrt[p]{a} k^*, \sqrt[p]{a} k^* \rangle = \Omega_{K/k^*}$.

Now to prove the isomorphism $H^2(L/k, \Omega_{k^*}) \cong H^2(K/k, \Omega_{k^*})$ in (ii), we shall refer to the proof of Theorem 9. Since $G(F/K)$ is cyclic of order $p^c$, invoking the computation of norm map before, we have

\[ N_{F/K}(\Omega_{k^*}) = \langle \varepsilon_{n_1}^{p^c} k^*, \varepsilon_{n_2}^{p^c} k^*, \sqrt[p]{a_1}^{p^c} k^*, \sqrt[p]{a_2}^{p^c} k^* \rangle \]

\[ = \langle \varepsilon_{n_1} k^*, \varepsilon_{n_2} k^*, \sqrt[p]{a_1} k^*, \sqrt[p]{a_2} k^* \rangle = \Omega_{k^*} \cong \langle \Omega_{k^*} \rangle_{G(F/K)} \]
for \( \gcd(p, n'_1) = \gcd(p, n'_2) = 1 \). Thus
\[
H^2(F/K, \Omega/k^*) = H^0(F/K, \Omega/k^*) = \frac{(\Omega/k^*)\hat{\Omega}(F/K)}{N_{F/K}(\Omega/k^*)} = 1,
\]
so that \( H^2(F/K, \Omega/k^*) = H^1(F/K, \Omega/k^*) = 1 \) due to the Herbrand quotient. Hence the exact sequence
\[
1 \rightarrow H^2(K/k, \Omega/k^*) \rightarrow H^2(F/k, \Omega/k^*) \xrightarrow{\text{res}} H^2(F/K, \Omega/k^*) = 1
\]
gives rise to the isomorphism \( H^2(K/k, \Omega/k^*) \cong H^2(F/k, \Omega/k^*) \).

Similarly, with the cyclic group \( \hat{G}(L/F) \) of order \( p^{n_2} \), we get
\[
N_{L/K}(\Omega/k^*) = \langle \varepsilon_{n_1}^{p^{n_1}}, \varepsilon_{n_2}^{p^{n_2}}, \sqrt[p]{a_1}^{p^{n_1}}, \sqrt[p]{a_2}^{p^{n_2}} \rangle k^* = \Omega/k^* = (\Omega/k^*)\hat{\Omega}(L/K),
\]
so \( H^1(L/F, \Omega/k^*) = H^2(L/F, \Omega/k^*) = H^0(L/F, \Omega/k^*) = 1 \). Thus the sequence
\[
1 \rightarrow H^2(F/k, \Omega/k^*) \rightarrow H^2(L/k, \Omega/k^*) \xrightarrow{\text{res}} H^2(L/F, \Omega/k^*) = 1
\]
yields an isomorphism \( H^2(F/k, \Omega/k^*) \cong H^2(L/k, \Omega/k^*) \).

\[\square\]

\textbf{Remark 3.} Due to Theorem 12, we now can generalize the reduction of cohomology on radical groups having finitely many \( n \)-th roots.

In Theorem 11, we furthermore assume that each \( \sqrt[p]{a_i} \) is a root of an irreducible binomial polynomial \( X^{n_i} - a_i \) in \( k[X] \). Then we can observe that the degrees \( [F : K] \) and \( [L : F] \) are exactly equal to \( p^{n_2} \) and \( p^{n_1} \) respectively. In fact, since \( X^{n_2} - a_2 \) is irreducible over \( k \), \( a_2 \) does not belong to \( k^r \) for all primes divisors \( r \) of \( n_2 \) due to [8, 16.6]. But since \( n_2 = p^{n_2}n'_2 \), \( a_2 \notin k^{p^{n_2}} \). Moreover it can be seen that \( \sqrt[p]{a_2} \notin \langle \sqrt[p]{a_1} \rangle \) for \( i = 1, 2 \). Thus \( \sqrt[p]{a_2} = \lambda_2 \) does not belong to \( k^p(\sqrt[p]{a_1}, \sqrt[p]{a_2}^{p^i}) = K^p \), so it follows from [8, 16.6] that \( X^{p^{n_2}} - \lambda_2 \) is irreducible over \( K \). Hence \( [F : K] = p^{n_2} \).

Similarly since \( L = F(\sqrt[p]{X^{n_1}}) \) where \( \sqrt[p]{X^{n_1}} \) is a root of \( X^{p^{n_2}} - \sqrt[p]{a_1} \in F[X] \), and \( X^{n_1} - a_1 \) is irreducible over \( k \), \( a_1 \notin k^p \) so \( a_1 \notin k^{p^{n_1}} \), i.e., \( \sqrt[p]{a_1} \notin k^p \). Clearly \( \sqrt[p]{a_1} \) does not belong to \( \langle \sqrt[p]{a_1} \rangle \) and \( \langle \sqrt[p]{a_2} \rangle \), so \( \lambda_1 = \sqrt[p]{a_1} \notin k^p(\sqrt[p]{a_1}, \sqrt[p]{a_2}) = F^p \), so \( X^{p^{n_1}} - \lambda_1 \) is irreducible over \( F \). Since \( \varepsilon_{p^{n_1}} \in F \), \( L/F \) is cyclic of degree \( p^{n_1} \).

5. Cohomological characterization of Brauer subgroups

We give our final observation with regard to Schur and radical subgroups of Brauer group. Let \( A \) be a Schur \( k \)-algebra. The set of similarity classes \([A] \) of \( A \) forms the \textit{Schur subgroup} \( S(k) \) of the Brauer group \( B(k) \). Let \( L \) be a finite Galois extension of \( k \). Then there is a restriction homomorphism \( S(k) \rightarrow S(L) \) defined by the tensor product \([A] \mapsto L \otimes_k [A] \) for \([A] \in S(k) \). The kernel \( S(L/k) \) of the homomorphism, called \textit{relative Schur group}, consists of Schur \( k \)-algebra classes split by \( L \). Analogously, the set of similarity classes of radical \( k \)-algebras forms the \textit{radical group} \( R(k) \). And for a finite Galois extension \( L \)
of $k$, the kernel of the restriction $R(k) \rightarrow R(L)$ is the relative radical group $R(L/k)$.

A well known theorem of Brauer-Witt provides an interpretation of Schur algebra as cyclotomic algebra, so $S(k)$ can be characterized cohomologically. An analog was conjectured in [2] that every projective Schur algebra is represented by a radical algebra so that a nice cohomological description can be provided on $PS(k)$. On the other hand, it has been verified cohomological characterizations for radical group in [2, 1.5] and for relative radical group in [5, Theorem 7].

**Theorem 13.** [5, Theorem 7] Let $L = k(\Omega)$ be a finite Galois radical extension of $k$. Then $R(L/k)$ is isomorphic to $H^2(L/k, \langle \epsilon_n \rangle)$ (Corollary 8 [5]). Moreover by employing Theorem 2, if $k \leq \mathbb{Q}(\epsilon_m)$ and $n$ and $n'$ are the same as in Theorem 2, then the following diagram is commutative:

$$
\begin{align*}
H^2(k(\epsilon_n')/k) & \xrightarrow{\inf} H^2(k(\epsilon_n)/k) \\
\downarrow \cong & \quad \downarrow \cong \\
S(k(\epsilon_n')/k) & \xrightarrow{= \sim} S(k(\epsilon_n)/k)
\end{align*}
$$

where all vertical and horizontal arrows are isomorphisms. This diagram provides a stronger relationship than that of Brauer and cohomology groups: for a Galois extension $k < L < E$, the diagram is commutative: (see [10, p.252], [11, p.159])

$$
\begin{align*}
H^2(L/k) & \xrightarrow{\inf} H^2(E/k) \\
\downarrow \cong & \quad \downarrow \cong \\
B(L/k) & \xrightarrow{=} B(E/k)
\end{align*}
$$

in which only vertical arrows are isomorphisms. Owing to Theorem 10, we obtain a diagram of radical and cohomology groups as following.

**Theorem 14.** Let $L = k(\Omega)$ and $L_0 = k(\Omega_0)$ be radical extensions of $k$ satisfying the same context as in Theorem 10. Then there is a homomorphism $\chi : R(L_0/k) \rightarrow R(L/k)$ that makes the following diagram commute.

$$
\begin{align*}
H^2(L_0/k, \Omega_0) & \xrightarrow{\chi} H^2(L/k, \Omega_0) \\
\downarrow & \quad \downarrow \\
H^2(L_0/k, \Omega) & \xrightarrow{=} H^2(L/k, \Omega)
\end{align*}
$$

$$
\begin{align*}
H^2(L_0/k) & \xrightarrow{\chi} H^2(L/k) \\
\downarrow \cong & \quad \downarrow \cong \\
R(L_0/k) & \xrightarrow{=} R(L/k)
\end{align*}
$$
Proof. We first note a difference here from (4) that \( H^2(L/k, \Omega) \to H^2(L/k, L^\ast) \) need not be one to one. Hence the vertical arrows \( H^2(L_0/k, \Omega_0) \to H^2_1(L_0/k) \) and \( H^2(L/k, \Omega) \to H^2_1(L/k) \) are only surjective homomorphisms.

The two vertical isomorphisms in the above diagram are due to Theorem 13. By Theorem 10, we have an isomorphism \( \psi : H^2(L_0/k, \Omega_0/k^\ast) \to H^2(L/k, \Omega_0/k^\ast) \). Moreover since the surjection \( \Omega_0 \to \Omega_0/k^\ast \) induces both homomorphisms \( H^2(L_0/k, \Omega_0) \to H^2(L_0/k, \Omega_0/k^\ast) \) and \( H^2(L/k, \Omega_0) \to H^2(L/k, \Omega_0/k^\ast) \), the homomorphism \( H^2(L_0/k, \Omega_0) \xrightarrow{\chi_1} H^2(L_0/k, \Omega_0/k^\ast) \) makes the diagram commute:

\[
\begin{array}{ccc}
H^2(L_0/k, \Omega_0) & \xrightarrow{\chi_1} & H^2(L_0/k, \Omega_0/k^\ast) \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
H^2(L_0/k, \Omega_0/k^\ast) & \cong & H^2(L/k, \Omega_0/k^\ast)
\end{array}
\]

Hence there exists a homomorphism \( \chi : R(L_0/k) \to R(L/k) \) which makes the diagram (5) commute. \( \square \)

A characterization of \( R(L/k) \) by means of cohomology was given in Theorem 13 that there is an isomorphism \( H^2_1(L/k) \cong R(L/k) \). An interesting cohomological description of radical group was proved in [2, Proposition 1.5] that if \( L = k_{rad} \) is the maximal radical extension of \( k \) in an algebraic closure \( \bar{k} \), then \( \mu(\bar{k}) \) is contained in \( L \) and there is a surjective homomorphism \( H^2(L/k, \mu) \to R(k) \). One may also refer to the cohomological characterization of \( PNil(k) \) in Proposition 1.6 [2] where \( PNil(k) \subset B(k) \) consist of classes that may be represented by a projective Schur algebras of nilpotent type. It would be interesting to discover any relationships between \( H^2(L/k, \Omega_0/k^\ast) \) and radical \( k \)-algebras split by \( L = k(\Omega_L) \).

References


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