MENDELSOHN TRIPLE SYSTEMS EXCLUDING CONTIGUOUS UNITS WITH $\lambda = 1$

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Abstract. We obtain a necessary and sufficient condition for the existence of Mendelsohn triple systems excluding contiguous units with $\lambda = 1$. Also, we obtain the spectrum for cyclic such systems.

1. Introduction

If $X = \{x_0, x_1, \ldots, x_{v-1}\}$ is a cyclically ordered set of points, then two points $x_i$ and $x_{i+1}$ are said to be contiguous points for all $i$ such that $0 \leq i \leq v-2$, as are $x_{v-1}$ and $x_0$. Otherwise, they are non-contiguous. A triple sampling plan excluding contiguous units $TSEC(v, \lambda)$ of order $v$ and index $\lambda$ is a pair $(X, \mathcal{B})$ where $X$ is a cyclically ordered $v$-set of points (units) and $\mathcal{B}$ is a collection of 3-subsets of $X$, called triples, such that any two contiguous points of $X$ do not appear in any triple while any two non-contiguous distinct points appear in exactly $\lambda$ triples of $\mathcal{B}$. When any two distinct points appears in precisely $\lambda$ triples of $\mathcal{B}$, it is a triple system $TS(v, \lambda)$. There exists a $TS(v, \lambda)$ if and only if $\lambda \equiv 0 \pmod{\gcd(v-2,6)}$ and $v \neq 2$ [8], and a $TSEC(v, \lambda)$ exists if and only if $v \in \{0, 3\}$ or $v \geq 9$ and $\lambda(v-3) \equiv 0 \pmod{6}$ [6].

Hung and Mendelsohn [10] considered triple systems in which the triples are ordered. A transitive triple $[x, y, z]$ is taken to contain the ordered pairs $(x, y), (x, z)$ and $(y, z)$. A directed triple system $DTS(v, \lambda)$ is a pair $(X, \mathcal{B})$ where $X$ is a $v$-set of points and $\mathcal{B}$ is a collection of transitive triples of $X$ such that every ordered pair $(x, y)$ of distinct points of $X$ appears in precisely $\lambda$ transitive triples in $\mathcal{B}$. Necessary and sufficient conditions for the existence of directed triple systems have been established by Hung and Mendelsohn for $\lambda = 1$ [10], and by Seberry and Skillicorn for all $\lambda$ [12]; they found that the conditions for the existence of a $DTS(v, \lambda)$ are the same as those for the existence of a $TS(v, 2\lambda)$. Mendelsohn [11] also considered triple systems in which the triples are ordered slightly different from transitive triples. A cyclic triple
(x, y, z) is a 3-set of the ordered pairs (x, y), (y, z) and (z, x). A Mendelsohn triple system MTS(v, λ) is a pair (X, B) where X is a v-set of points and B is a collection of cyclic triples of X such that every ordered pair (x, y) of distinct points of X appears in precisely λ cyclic triples in B. The existence of a MTS(v, λ) was settled by Mendelsohn for λ = 1 [11], and by Bennett for all λ ≥ 2 [2]. There exists a DT S(6, 1) while there does not exists a MTS(6, 1).

Analogously, if X is a cyclically ordered v-set of points, then we may define a directed [Mendelsohn] sampling plan excluding contiguous units DT SEC(v, λ) [MT SEC(v, λ)] based on X as a collection B of transitive [cyclic] triples of X such that any ordered pair of two contiguous points of X does not appear in any transitive [cyclic, respectively] triple in B while each ordered pair of two non-contiguous distinct points appears in exactly λ members of B.

Naturally, if we treat the transitive [cyclic] triples of a DTS(v, λ) [MTS(v, λ)] as unordered 3-subsets, we obtain a TS(v, 2λ) which is called the underlying triple system of the DTS(v, λ) [MTS(v, λ), respectively]. The underlying triple system of a DTS(v, λ) [MTS(v, λ)] is termed directable [cyclable, respectively].

**Theorem 1.1** ([4, 5, 9]). Every TS(v, 2λ) is directed.

From Theorem 1.1, every TSEC(v, 2λ) is obviously directed, and hence the existence of a DT SEC(v, λ) is equivalent to the existence of a TSEC(v, 2λ). From [6], we have the following theorem.

**Theorem 1.2.** There exists a DT SEC(v, λ) if and only if v ∈ {0, 3} or v ≥ 9 and λv ≡ 0 (mod 3).

There exists a TS(6, 2), but not a MTS(6, 1). Thus not every TS(v, 2λ) is cyclic and hence every TSEC(v, λ) is not cyclical. Therefore, the existence of a MT SEC(v, λ) is in doubt. In this paper, we obtain a necessary and sufficient condition for the existence of a MT SEC(v, λ) with λ = 1. We simply denote MT SEC(v) for MT SEC(v, 1). Since the existence of a MT SEC(v, λ) implies the existence of a TSEC(v, 2λ), we have the following necessary condition.

**Lemma 1.3.** If there exists a MT SEC(v, λ), then v ∈ {0, 3} or v ≥ 9 and λv ≡ 0 (mod 3). When λ = 1, it is v ≡ 0 (mod 3) and v ≠ 6.

In the next section, we will show that there exists a MT SEC(v) for all v ≡ 0 (mod 3) and v ≠ 6 (we will omit the trivial case v ∈ {0, 3}).

2. The existence of MT SEC(v)’s

We will first construct a MT SEC(v) for v = 12, 18, 24, 30, 36, 42, 48, 60 and 78, and then establish the existence of a MT SEC(v) by employing a recursive method for all v ≡ 0 (mod 3) and v ≠ {0, 3, 6}. An automorphism of a MT SEC(v), (X, B), is a permutation α on X such that each cyclic triple of B maps onto a cyclic triple. A MT SEC(v) is said to be bicyclic if it admits
an automorphism consisting of exactly two cycles of length \( \frac{v}{2} \). Such an
automorphism is said to be bicyclic. If \((X, \mathcal{B})\) is a bicyclic MTSEC\((v)\) with a
bicyclic automorphism \(\alpha\), then the group \(\langle \alpha \rangle\) generated by \(\alpha\) acts on \(\mathcal{B}\). Thus
\(\mathcal{B}\) is partitioned into mutually disjoint orbits. A collection of cyclic triples,
called base blocks, which are taken each of the orbits exactly once represents the
bicyclic MTSEC\((v)\) together with the bicyclic automorphism. In order to
avoid the presentation of many large systems, whenever possible, we present a
collection of base blocks for a bicyclic MTSEC\((v)\). It is convenient to name
the points as ordered pairs from

\[
\left\{ 0, 1, \ldots, \frac{v-2}{2} \right\} \times \{1, 2\},
\]

taking \((i, 1)\) and \((j, 2)\) to be contiguous when \(i = j\), or \(i \equiv j + 1 \pmod{\frac{v}{2}}\). We
write shortly \(x_i\) for the ordered pair \((x, i)\).

We will construct that our bicyclic MTSEC\((v)\) is based on \(\{0, 1, \ldots, \frac{v-2}{2}\} \times \{1, 2\}\) and the corresponding bicyclic automorphism is

\[
\left(0_1, 1_1, \ldots, \frac{v}{2} - 1\right) \left(0_2, 1_2, \ldots, \frac{v}{2} - 1\right).
\]

**Lemma 2.1.** There exists a bicyclic MTSEC\((v)\) for \(v = 12, 18, 24, 30, 36, 42, 48, 60\) and 78.

**Proof.** A collection of base blocks for a MTSEC\((v)\) is

when \(v = 12:\)

\[
\begin{align*}
(1_1, 0_1, 4_2), & \quad (0_2, 1_2, 5_1), \quad (0_1, 4_1, 1_2), \quad (0_2, 2_1, 4_2), \quad (0_1, 2_1, 5_1), \\
(0_2, 5_2, 2_2),
\end{align*}
\]

when \(v = 18:\)

\[
\begin{align*}
(0_1, 1_1, 4_1), & \quad (0_1, 4_1, 1_1), \quad (0_2, 1_2, 3_2), \quad (0_1, 2_1, 7_2), \quad (7_2, 2_1, 0_1), \\
(4_2, 3_2, 0_1), & \quad (3_2, 1_2, 0_1), \quad (0_1, 1_2, 4_2), \quad (2_2, 6_2, 0_1), \quad (6_2, 2_2, 0_1),
\end{align*}
\]

when \(v = 24:\)

\[
\begin{align*}
(0_1, 1_1, 10_1), & \quad (0_1, 3_1, 8_1), \quad (0_1, 6_1, 2_1), \quad (0_2, 1_2, 5_2), \quad (1_1, 0_1, 9_2), \\
(5_1, 0_1, 10_2), & \quad (8_2, 7_2, 0_1), \quad (7_2, 3_2, 0_1), \quad (3_2, 1_2, 0_1), \quad (1_2, 6_2, 0_1), \\
(6_2, 9_2, 0_1), & \quad (2_2, 4_2, 0_1), \quad (4_2, 10_2, 0_1), \quad (5_2, 2_2, 0_1),
\end{align*}
\]

when \(v = 30:\)

\[
\begin{align*}
(0_1, 1_1, 4_1), & \quad (0_1, 4_1, 1_1), \quad (0_1, 2_1, 8_1), \quad (0_1, 8_1, 2_1), \quad (1_2, 0_2, 4_2), \\
(0_1, 5_1, 10_2), & \quad (5_1, 0_1, 10_2), \quad (2_2, 3_2, 0_1), \quad (3_2, 9_2, 0_1), \quad (9_2, 11_2, 0_1), \\
(11_2, 6_2, 0_1), & \quad (6_2, 13_2, 0_1), \quad (13_2, 7_2, 0_1), \quad (7_2, 12_2, 0_1), \quad (12_2, 8_2, 0_1), \\
(8_2, 1_2, 0_1), & \quad (1_2, 4_2, 0_1), \quad (4_2, 2_2, 0_1),
\end{align*}
\]
when \( v = 36 \):

\[
\begin{align*}
(0_1, 1_1, 15_1), & \quad (0_1, 2_1, 12_1), & \quad (0_1, 9_1, 3_1), & \quad (0_1, 7_1, 5_1), & \quad (0_1, 8_1, 1_1), \\
(0_1, 4_1, 16_2), & \quad (0_1, 5_1, 7_2), & \quad (0_2, 1_2, 9_2), & \quad (6_2, 5_2, 0_1), & \quad (5_2, 11_2, 0_1), \\
(11_2, 4_2, 0_1), & \quad (4_2, 9_2, 0_1), & \quad (9_2, 1_2, 0_1), & \quad (1_2, 8_2, 0_1), & \quad (8_2, 2_2, 0_1), \\
(7_2, 3_2, 0_1), & \quad (3_2, 6_2, 0_1), & \quad (16_2, 13_2, 0_1), & \quad (13_2, 15_2, 0_1), & \quad (15_2, 10_2, 0_1), \\
(10_2, 14_2, 0_1), & \quad (14_2, 12_2, 0_1).
\end{align*}
\]

when \( v = 42 \):

\[
\begin{align*}
(0_1, 1_1, 5_1), & \quad (0_1, 5_1, 1_1), & \quad (0_1, 2_1, 10_1), & \quad (0_1, 10_1, 2_1), & \quad (0_1, 3_1, 9_1), \\
(0_1, 9_1, 3_2), & \quad (0_1, 7_1, 14_2), & \quad (7_1, 0_1, 14_2), & \quad (1_2, 0_2, 4_2), & \quad (11_2, 19_2, 0_1), \\
(19_2, 11_2, 0_1), & \quad (4_2, 6_2, 0_1), & \quad (6_2, 4_2, 0_1), & \quad (3_2, 8_2, 0_1), & \quad (8_2, 3_2, 0_1), \\
(2_2, 9_2, 0_1), & \quad (9_2, 2_2, 0_1), & \quad (13_2, 16_2, 0_1), & \quad (16_2, 17_2, 0_1), & \quad (17_2, 13_2, 0_1), \\
(12_2, 10_2, 0_1), & \quad (10_2, 1_2, 0_1), & \quad (5_2, 15_2, 0_1), & \quad (15_2, 5_2, 0_1), & \quad (12, 18_2, 0_1), \\
(18_2, 12_2, 0_1), & \quad (18_2, 12_2, 0_1).
\end{align*}
\]

when \( v = 48 \):

\[
\begin{align*}
(0_1, 1_1, 3_1), & \quad (0_1, 3_1, 1_1), & \quad (0_1, 13_1, 4_1), & \quad (0_1, 12_1, 5_1), & \quad (0_1, 14_1, 6_1), \\
(0_1, 4_1, 13_1), & \quad (0_1, 6_1, 14_1), & \quad (0_1, 5_1, 12_2), & \quad (0_1, 7_1, 19_2), & \quad (0_2, 1_2, 12_2), \\
(19_2, 22_2, 0_1), & \quad (22_2, 20_2, 0_1), & \quad (20_2, 15_2, 0_1), & \quad (15_2, 17_2, 0_1), & \quad (17_2, 16_2, 0_1), \\
(16_2, 21_2, 0_1), & \quad (21_2, 18_2, 0_1), & \quad (18_2, 7_2, 0_1), & \quad (12, 11_2, 0_1), & \quad (11_2, 12, 0_1), \\
(2_2, 10_2, 0_1), & \quad (10_2, 2_2, 0_1), & \quad (3_2, 9_2, 0_1), & \quad (9_2, 3_2, 0_1), & \quad (4_2, 8_2, 0_1), \\
(8_2, 42, 0_1), & \quad (5_2, 14_2, 0_1), & \quad (14_2, 5_2, 0_1), & \quad (6_2, 13_2, 0_1), & \quad (13_2, 6_2, 0_1), \\
(8_2, 42, 0_1), & \quad (5_2, 14_2, 0_1), & \quad (14_2, 5_2, 0_1), & \quad (6_2, 13_2, 0_1), & \quad (13_2, 6_2, 0_1), \\
\end{align*}
\]

when \( v = 60 \):

\[
\begin{align*}
(0_1, 1_1, 25_1), & \quad (0_1, 15_1, 2), & \quad (0_1, 3_1, 22_1), & \quad (0_1, 4_1, 16_1), & \quad (0_1, 6_1, 17_1), \\
(0_1, 27_1, 7_1), & \quad (0_1, 18_1, 8_1), & \quad (0_1, 26_1, 5_1), & \quad (1_1, 0_1, 27_2), & \quad (0_1, 21_2, 28_2), \\
(0_1, 7_1, 23_2), & \quad (9_1, 0_1, 12_2), & \quad (14_1, 0_1, 22_2), & \quad (0_2, 15_2, 14_2), & \quad (0_2, 12_2, 9_2), \\
(24_2, 16_2, 0_1), & \quad (23_2, 17_2, 0_1), & \quad (17_2, 21_2, 0_1), & \quad (21_2, 27_2, 0_1), & \quad (28_2, 24_2, 0_1), \\
(5_2, 10_2, 0_1), & \quad (10_2, 5_2, 0_1), & \quad (4_2, 11_2, 0_1), & \quad (11_2, 4_2, 0_1), & \quad (3_2, 12_2, 0_1), \\
(2_2, 13_2, 0_1), & \quad (13_2, 2_2, 0_1), & \quad (6_2, 9_2, 0_1), & \quad (9_2, 6_2, 0_1), & \quad (8_2, 22_2, 0_1), \\
(19_2, 7_2, 0_1), & \quad (7_2, 19_2, 0_1), & \quad (25_2, 15_2, 0_1), & \quad (15_2, 25_2, 0_1), & \quad (14_2, 12_2, 0_1), \\
(12_2, 14_2, 0_1), & \quad (20_2, 18_2, 0_1), & \quad (18_2, 20_2, 0_1), & \quad (18_2, 20_2, 0_1).
\end{align*}
\]
when \( v = 78 \):

\[
(0_1, 11, 17_1), \quad (0_1, 17_1, 11_1), \quad (0_1, 2_1, 20_1), \quad (0_1, 20_1, 2_1), \quad (0_1, 3_1, 11_1),
\]
\[
(0_1, 11_1, 3_1), \quad (0_1, 4_1, 8_1), \quad (0_1, 8_1, 4_1), \quad (0_1, 5_1, 12_1), \quad (0_1, 12_1, 5_1),
\]
\[
(0_1, 6_1, 15_1), \quad (0_1, 15_1, 6_1), \quad (0_1, 13_1, 26_2), \quad (13_1, 0_1, 26_2), \quad (1_2, 0_2, 4_2),
\]
\[
(25_2, 31_2, 0_1), \quad (31_2, 25_2, 0_1), \quad (29_2, 32_2, 0_1), \quad (32_2, 37_2, 0_1), \quad (37_2, 33_2, 0_1),
\]
\[
(33_2, 28_2, 0_1), \quad (28_2, 36_2, 0_1), \quad (36_2, 34_2, 0_1), \quad (34_2, 35_2, 0_1), \quad (35_2, 27_2, 0_1),
\]
\[
(27_2, 29_2, 0_1), \quad (1_2, 20_2, 0_1), \quad (20_2, 1_2, 0_1), \quad (2_2, 19_2, 0_1), \quad (19_2, 2_2, 0_1),
\]
\[
(3_2, 18_2, 0_1), \quad (18_2, 3_2, 0_1), \quad (4_2, 17_2, 0_1), \quad (17_2, 4_2, 0_1), \quad (5_2, 16_2, 0_1),
\]
\[
(16_2, 5_2, 0_1), \quad (6_2, 15_2, 0_1), \quad (15_2, 6_2, 0_1), \quad (7_2, 14_2, 0_1), \quad (14_2, 7_2, 0_1),
\]
\[
(8_2, 24_2, 0_1), \quad (24_2, 8_2, 0_1), \quad (9_2, 23_2, 0_1), \quad (23_2, 9_2, 0_1), \quad (10_2, 22_2, 0_1),
\]
\[
(22_2, 10_2, 0_1), \quad (11_2, 21_2, 0_1), \quad (21_2, 11_2, 0_1), \quad (12_2, 30_2, 0_1), \quad (30_2, 12_2, 0_1).
\]

\[\square\]

**A group divisible design (GDD) of order** \( v \) **and index** \( \lambda \) **is a triple** \((X, G, \mathcal{B})\) **which satisfies the following properties:**

1. \( X \) is a \( v \)-set of points,
2. \( G \) is a partition of \( X \) whose members are called **groups**, and
3. \( \mathcal{B} \) is a collection of subsets of \( X \), called **blocks**, such that any block and any group contain at most one common point, and every pair of points from distinct groups occurs in exactly \( \lambda \) blocks.

The **group-type** (type) of the GDD is the multiset \( \{|G| : G \in G\} \). We use the notation for group-type: \( g_1^{u_1}g_2^{u_2} \cdots g_s^{u_s} \) indicates that there are \( u_i \) groups of size \( g_i \) for \( 1 \leq i \leq s \). The set \( K = \{|B| : B \in \mathcal{B}\} \) is the set of block sizes of the GDD, and the notation \( K\text{-GDD} \) is used to denote a GDD whose block sizes lie in the set \( K \). When \( K = \{k\} \), we write \( k\text{-GDD} \) for \( \{k\}\text{-GDD} \).

A **Latin square of side** \( n \) **is an** \( n \times n \) **array based on a set** \( S \) **of** \( n \) **symbols with the property that every row and every column contains every symbol exactly once. Two Latin squares** \( A = (a_{ij}) \) **and** \( B = (b_{ij}) \) **of the same side** \( n \) **are said to be orthogonal if the** \( n^2 \) **ordered pairs** \((a_{ij}, b_{ij})\) **, the pairs formed superimposing one square on the other, are all different. There exist three mutually orthogonal Latin squares of side** \( n \) **for all** \( n \neq 2, 6, 10 \) \([13,14]\). The existence of a 4-GDD of** \( n^2 \) **and index 1 is equivalent to the existence of three mutually orthogonal Latin squares of side** \( n \) \([1]\). Thus there exists a 4-GDD of** \( n^2 \) **and index 1 for all** \( n \) **except for** \( n = 2, 3, 6, 10 \).

A **3-GDD** with index \( \lambda \), \((X, G, \mathcal{B})\), **is called a Mendelsohn group divisible design (MGDD)** **if each block of** \( \mathcal{B} \) **is considered as a cyclic triple and every ordered pair of points from distinct groups occurs in exactly \( \lambda \) blocks. It is not hard to construct a MGDD of type** \( 2^i \), \( 3 \leq i \leq 4 \), **and index 1. Namely, taking groups** \( \{1, 2\}, \{3, 4\}, \{5, 6\} \) **and cyclic triples**

\[
(1, 3, 5), (1, 5, 3), (1, 4, 6), (1, 6, 4), (2, 3, 6), (2, 6, 3), (2, 4, 5), (2, 5, 4)
\]
we have a MGDD of type $2^i$ and index $1$, and groups \{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\} and cyclic triples
\[
(1, 3, 6), \ (1, 4, 8), \ (1, 5, 7), \ (2, 3, 7), \ (2, 4, 5), \ (2, 6, 8), \ (3, 5, 8), \ (4, 6, 7), \ (1, 6, 3), \ (1, 8, 4), \ (1, 7, 5), \ (2, 7, 3), \ (2, 5, 4), \ (2, 8, 6), \ (3, 8, 5), \ (4, 7, 6)
\]
form a MGDD of type $2^4$ and index $1$. Consequently, there exists a MGDD of type $2^i$, $3 \leq i \leq 4$, and index $\lambda \geq 1$.

A partial triple system $PTS(v, \lambda)$ of order $v$ and index $\lambda$ is a pair $(V, \mathfrak{B})$ where $V$ is a $v$-set of points and $\mathfrak{B}$ is a collection of $3$-subsets of $V$, called triples, such that every $2$-subset of $V$ appears in at most $\lambda$ triples of $\mathfrak{B}$. The leave of a $PTS(v, \lambda)$ is the collection of pairs of points, which appear fewer than $\lambda$ times in the triples of the $PTS(v, \lambda)$ and if a pair $\{x, y\}$ appears in $s(\leq \lambda)$ triples then it appears $\lambda - s$ times in the leave. Let $V = \{x_1, x_2, \ldots, x_{2m+1}\}$ be a $(2m + 1)$-set. If $2m + 1 = 1, 5$ (mod $6$), then Colbourn and Rosa \cite{7} show that there exists a $PTS(2m + 1, 1)$ whose leave is the set $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{2m-1}, x_{2m}\}, \{x_{2m}, x_1\}\}$. The following lemma is slightly modified the Lemma 3.1 of Colbourn and Ling \cite{6}.

**Lemma 2.2.** (1) Let $m \neq 2, 3, 6, 10$ and $x = 0$ or $5 \leq x \leq m$. If there exist both a $MTSEC(2m, \lambda)$ and a $MTSEC(2x, \lambda)$, then there exists a $MTSEC(6m + 2x, \lambda)$.

(2) Let $m \neq 2, 3, 6, 10$, $4 \leq x \leq m$ and $2m + 1 \equiv 1, 5$ (mod $6$). If there exists a $MTSEC(2x + 1, \lambda)$, then there exists a $MTSEC(6m + 2x + 1, \lambda)$.

**Proof.** Since $m \neq 2, 3, 6, 10$, there exists a $4$-GDD of type $m^4$ and index $1$. Let us have a $4$-GDD of type $m^4$ and index $1$, whose groups are $G_1, G_2, G_3$ and $G_4$, and blocks $\mathfrak{B}$. Partition $G_4$ into two disjoint subsets $A$ and $B$ so that $|A| = x$, and $|A| > 0$ whenever $x > 0$. For Case (1), a $MTSEC(6m + 2x, \lambda)$ to be constructed has points
\[
(G_1 \cup G_2 \cup G_3 \cup A) \times \{1, 2\},
\]
and in Case (2), a $MTSEC(6m + 2x + 1, \lambda)$ to be constructed has the same points together with an additional point $\infty$.

Choose one block $D \in \mathfrak{B}$, so that when $A$ is nonempty, $D \cap A \neq \emptyset$ (when $A = \emptyset$, one has $x = 0$; in this case choose any block to serve as $D$).

For each block $\{u, v, w, z\} \in \mathfrak{B}$ other than $D$ with $z \in G_4$, set
\[
\sigma = 0 \text{ if } z \in B; \text{ and } \sigma = 2 \text{ if } z \in A.
\]
Then form a MGDD of type $2^i\sigma^j$ and index $\lambda$ with groups
\[
\{(w, 1), (w, 2)\}, \quad \{(v, 1), (v, 2)\},
\]
\[
\{(w, 1), (w, 2)\}, \quad \{(z, i) | i = 1, \sigma\} (= \emptyset \text{ if } \sigma = 0).
\]

Next, we handle the block $D = \{a_1, a_1, a_1, a_1\}$ with $a_{i_1} \in G_i$, $i = 1, 2, 3, 4$. For $i = 1, 2, 3$, we assume that
\[
G_i \times \{1, 2\} = \{(a_{i_1}, 1), (a_{i_1}, 2), (a_{i_2}, 1), (a_{i_2}, 2), \ldots, (a_{i_m}, 1), (a_{i_m}, 2)\}
\]
is cyclically ordered, and if \( a_{14} \in A \), then
\[
A \times \{1, 2\} = \{(a_{14}, 1), (a_{14}, 2), (a_{24}, 1), (a_{24}, 2), \ldots, (a_{x4}, 1), (a_{x4}, 2)\}
\]
is assumed to be cyclically ordered for Case (1). In Case (2),
\[
A \times \{1, 2\} \cup \{\infty\} = \{(a_{14}, 1), (a_{14}, 2), (a_{24}, 1), (a_{24}, 2), \ldots, (a_{x4}, 1), (a_{x4}, 2), \infty\}
\]
is assumed to be cyclically ordered. If \( a_{14} \in B \), form a \( MGDD \) of type \( 2^3 \) and
index \( \lambda \) with groups
\[
\{(a_{m1}, 2), (a_{12}, 1)\}, \{(a_{m2}, 2), (a_{13}, 1)\}, \{(a_{m3}, 2), (a_{11}, 1)\}.
\]
If \( a_{14} \notin B \), then, by assumption, \( a_{14} \in A \). In Case (1), place the blocks of a
\( MGDD \) of type \( 2^4 \) and index \( \lambda \) with groups
\[
\{(a_{m1}, 2), (a_{12}, 1)\}, \{(a_{m2}, 2), (a_{13}, 1)\}, \{(a_{m3}, 2), (a_{14}, 1)\}, \{(a_{x4}, 2), (a_{11}, 1)\};
\]
in Case (2), with groups
\[
\{(a_{m1}, 2), (a_{12}, 1)\}, \{(a_{m2}, 2), (a_{13}, 1)\}, \{(a_{m3}, 2), (a_{14}, 1)\}, \{(\infty, (a_{11}, 1))\}.
\]

We now treat the cases separately for the rest, observing Case (1) first. For
\( i = 1, 2, 3 \), form a \( MTSEC(2m, \lambda) \) on the points \( G_i \times \{1, 2\} \) with the given cyclically ordering.

Next, if \( x > 0 \), form a \( MTSEC(2r, \lambda) \) on the points \( A \times \{1, 2\} \) with the given cyclically ordering. The resulting cyclic triples form a required \( MTSEC(6m + 2r, \lambda) \).

Let us turn to Case (2). For \( i = 1, 2, 3 \), form a \( PTS(2m + 1, 1) \) on the points \( (G_i \times \{1, 2\}) \cup \{\infty\} \) whose leave is the cycle of \( 2m \) points of \( G_i \times \{1, 2\} \), that is,
\[
\{(a_{i1}, 1), (a_{i1}, 2)\}, \{(a_{i2}, 2), (a_{21}, 1)\}, \{(a_{21}, 1), (a_{22}, 2)\}, \ldots,
\]
\[
\{(a_{mi}, 1), (a_{mi}, 2)\}, \{(a_{mi}, 2), (a_{1i}, 1)\},
\]
then form \( \lambda \) times cyclic triples \( (a, b, c) \) and \( (a, c, b) \) each for each block \( \{a, b, c\} \) of the \( PTS(2m + 1, 1) \).

Next, form a \( MTSEC(2r + 1, \lambda) \) on the points \( A \times \{0, 1\} \cup \{\infty\} \) with the given cyclically ordering. The resulting cyclic triples form a required \( MTSEC(6m + 2r + 1, \lambda) \). \( \square \)

If we replace each triple \( \{x, y, z\} \) of a \( TSEC(v, 1) \) by two cyclic triples
\( \{x, y, z\} \) and \( \{x, z, y\} \), the resulting cyclic triples form a \( MTSEC(v) \). Since there exists a \( TSEC(v, 1) \) for \( v \equiv 3 \pmod{6} \) \( [6] \), so does a \( MTSEC(v) \) for such all \( v \). Thus we have the following lemma.

**Lemma 2.3.** If \( v \equiv 3 \pmod{6} \), then there exists a \( MTSEC(v) \).

**Lemma 2.4.** If \( v \equiv 0 \pmod{6} \) and \( v \neq 6 \), then there exists a \( MTSEC(v) \).

**Proof.** Let \( v \equiv 0 \pmod{6} \) and \( v \neq 6 \). By Lemma 2.1, there exists a \( MTSEC(v) \) for \( v = 12, 18, 24, 30, 36, 42, 48, 60, 78 \). Now, Lemma 2.2 is applied to be existing of a \( MTSEC(v) \) for \( v = 54, 66, 72, 84 \). Write \( v = 6m + 2r \) where \( m \equiv 0 \).
(mod 3), \( m \geq 12 \) and \( x \in \{0, 12, 6\} \). Then, by Lemma 2.2, there exists a MTSEC(v).

Lemmas 1.3, 2.3 and 2.4 together yield the following theorem.

**Theorem 2.5.** There exists a MTSEC(v) if and only if \( v \equiv 0 \) (mod 3) and \( v \neq 6 \).

## 3. Concluding remarks

A TSEC(v, \( \lambda \)) is said to be cyclic if it admits an automorphism consisting of a single cycle of length \( v \). Wei [14] shows that there exists a cyclic TSEC(v, 1) if and only if \( v \equiv 3 \) (mod 6), and a cyclic TSEC(v, 2) exists if and only if \( v \equiv 0, 3, 9 \) (mod 12). Colbourn [3] shows that every cyclic DTSEC(v, \( 2\lambda \)) is directable, so there exists a cyclic TSEC(v, 1) if and only if \( v \equiv 0, 3, 9 \) (mod 12).

Let \((a, b, c)\) be a base block of a cyclic MTSEC(v, 1) based on the cyclically ordered set \( \{0, 1, \ldots, v-1\} \). Define the difference triple \([x, y, z]\) corresponding to \((a, b, c)\) so that

\[
\begin{align*}
  x &\equiv b - a \pmod{v}, \\
  y &\equiv c - b \pmod{v}, \\
  z &\equiv a - c \pmod{v}.
\end{align*}
\]

Then we see that the existence of a cyclic MTSEC(v, 1) is equivalent to partition of the set \( \{2, 3, \ldots, v-2\} \) into disjoint difference triples \([x, y, z]\) such that \( x + y + z \equiv 0 \pmod{v} \). Thus, if there exists a cyclic MTSEC(v, 1), then

\[
\sum_{i=2}^{v-2} i = \frac{v(v-3)}{2} \equiv 0 \pmod{v}, \text{ equivalently, } v - 3 \equiv 0 \pmod{2}.
\]

Thus \( v \) cannot be even. Therefore, if there exists a cyclic MTSEC(v), then \( v \equiv 3 \) (mod 6) since \( v \) must be odd and \( v \equiv 0 \) (mod 3) and \( v \neq 6 \). Since there exists a cyclic TSEC(v, 1) for \( v \equiv 3 \) (mod 6) [15], we have the following theorem.

**Theorem 3.1.** There exists a cyclic MTSEC(v, 1) if and only if \( v \equiv 3 \) (mod 6).

By Lemma 1.3, a necessary condition for the existence of a MTSEC(v, \( \lambda \)) is

\[
\lambda \equiv 1, 2 \pmod{3} \quad \text{and} \quad v \equiv 0 \pmod{3}, v \neq 6, \quad \text{or}
\]
\[
\lambda \equiv 0 \pmod{3} \quad \text{and} \quad v \in \{0, 3\} \quad \text{or} \quad v \geq 9.
\]

Since the union of a MTSEC(v, \( \lambda_1 \)) and a MTSEC(v, \( \lambda_2 \)) is a MTSEC(v, \( \lambda_1 + \lambda_2 \)), it suffices to establish the existence of a MTSEC(v, \( \lambda \)) for the minimum value of \( \lambda \), namely for

\[
\lambda = 1 \quad \text{and} \quad v \equiv 0 \pmod{3}, v \neq 6, \quad \text{or}
\]
\[
\lambda = 3 \quad \text{and} \quad v \equiv 1, 2 \pmod{3}, v \geq 9.
\]

To complete the existence of a MTSEC(v, \( \lambda \)) for all \( \lambda \), we need the existence of a MTSEC(v, 3) for \( v \equiv 1, 2 \pmod{3} \) with \( 10 \leq v \leq 50 \) which is unsettled and then Lemma 2.2 is applied.
References


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