SOME RESULTS ON ASYMPTOTIC BEHAVIORS OF RANDOM SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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Let \( \{X_n, n \geq 1\} \) be a sequence of independent identically distributed (i.i.d.) random variables (r.vs.), defined on a probability space \((\Omega, \mathcal{A}, P)\), and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued r.vs., defined on the same probability space \((\Omega, \mathcal{A}, P)\). Furthermore, we assume that the r.vs. \( N_n, n \geq 1 \) are independent of all r.vs. \( X_n, n \geq 1 \).

In present paper we are interested in asymptotic behaviors of the random sum

\[
S_{N_n} = X_1 + X_2 + \cdots + X_{N_n}, \quad S_0 = 0,
\]

where the r.vs. \( N_n, n \geq 1 \) obey some defined probability laws. Since the appearance of the Robbins’s results in 1948 ([8]), the random sums \( S_{N_n} \) have been investigated in the theory probability and stochastic processes for quite some time (see [1], [4], [2], [3], [5]).

Recently, the random sum approach is used in some applied problems of stochastic processes, stochastic modeling, random walk, queue theory, theory of network or theory of estimation (see [10], [12]).

The main aim of this paper is to establish some results related to the asymptotic behaviors of the random sum \( S_{N_n} \), in cases when the \( N_n, n \geq 1 \) are assumed to follow concrete probability laws as Poisson, Bernoulli, binomial or geometry.

1. Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of independent identically distributed (i.i.d.) random variables (r.vs.), defined on a probability space \((\Omega, \mathcal{A}, P)\) and let \( \{N_n, n \geq 1\} \) be a sequence of positive integer-valued r.vs., defined on the same probability space \((\Omega, \mathcal{A}, P)\). Furthermore, we assume that the r.vs. \( N_n, n \geq 1 \) are independent of all i.i.d.r.vs. \( X_n, n \geq 1 \). From now on, the random sum is defined by

\[
S_{N_n} = X_1 + X_2 + \cdots + X_{N_n}, \quad S_0 = 0.
\]
Since the appearance of the Robbins’s results in 1948 (see [8] for more details), the random sums $S_{N_n}$ have been investigated in the theory probability and stochastic processes for quite some time (see [5], [1], [2] and [12] for the complete bibliography).

In the classical theory of limit theorems we can consider the non-random index $n$ in the sum $S_n = X_1 + X_2 + \cdots + X_n$ as a random variable degenerated at a point $n$. Therefore, the replacement of the number $n$ of the sum $S_n$ by the positive integer-valued r.v.s. is natural. In simple terms, a random sum is a sum of a random number of r.v.s. The number of the terms $N_n$, $n \geq 1$ in the sum, as well as the individual terms, can obey various probability laws. In stochastic theory, $N_n$, $n \geq 1$ are often assumed to follow Poisson law or geometric law. In general, the r.v.s. $N_n$, $n \geq 1$ should satisfy any conditions. To illustrate this, we can recall three types of classical conditions for the r.v.s. $N_n$ as follows

\[(2)\quad E(N_n) \to +\infty \quad \text{as} \quad n \to \infty,\]

\[(3)\quad \frac{N_n}{n} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty,\]

or

\[(4)\quad N_n \xrightarrow{p} \infty \quad \text{as} \quad n \to \infty.\]

The condition (2) was used in well-known results of H. Robbins’s in 1948 (see details in [8]). The condition (3) was applied in Feller’s theorem for random sums (cf. [1]), while the last condition in (4) was presented in various papers like [4], [5], [9], [6], [7], . . . . It is to be noticed that the conditions (2), (3) and (4) are in following relationship

\[(3) \Rightarrow (4) \Rightarrow (2).\]

This paper is organized as follows. In Section 2 we present the main results related to the asymptotic behaviors of random sums in (1), when the r.v.s. $N_n$, $n \geq 1$ belong to some discrete probability laws. The proofs of these main results are presented in Section 3.

2. Main results

From now on, the random variable of standard normal law $N(0,1)$ will be denoted by $X^*$, the notation $\overset{d}{\to}$ will mean the convergence in distribution and $\overset{p}{\to}$ will denote the convergence in probability.

**Theorem 2.1.** Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d r.v.s., $X_j \sim \text{Bernoulli}(p)$, $p \in (0,1)$, $j = 1,2,\ldots, n$. Moreover, suppose that $\{N_n, n \geq 1\}$ is a sequence of positive integer-valued r.v.s. independent on all $X_j, j = 1,2,\ldots, n$. Then,

1. $S_{N_n} \sim \text{Poisson}(\lambda p)$, if $N_n \sim \text{Poisson}(\lambda), \lambda > 0, \ n \geq 1$.
2. $S_{N_n} \sim \text{Geometry}(q_{p+q-pq})$, if $N_n \sim \text{Geometry}(q), q \in (0,1), \ p + q = 1, \ n \geq 1$ with $\Pr(N_n = k) = q(1-q)^k, k = 0,1,\ldots$. 
(iii) \( S_{N_n} \sim Binomial(n, pq) \), if \( N_n \sim Binomial(n, q) \), \( q \in (0, 1) \), \( p + q = 1 \), \( n \geq 1 \).

It is well known that the sum of independent r.vs. from Bernoulli law will belong to the same law. But in cases (i) and (ii) the random sums of independent r.vs. of Bernoulli law will not obey the Bernoulli. The final distributions of the random sums \( S_{N_n} \) depend on the distribution of r.vs. \( N_n \), \( n \geq 1 \).

The same concludes would be found in B. Gnedenko’s or V. Kruglov’s and V. Korolev’s papers, in cases when the r.vs. \( X_j, j = 1, 2, \ldots, n \) were independent standard normal distributed while the r.vs. \( N_n, n \geq 1 \) were uniformly distributed (see for more details in [2], [3] and [5]). The conclusion in (iii) is a very interesting result when the random sum \( S_{N_n} \) and the r.vs. \( N_n, n \geq 1 \) are identically distributed. Furthermore, we can receive the Poisson’s Approximation Theorem by using Theorem 2.1(i) as follows

**Theorem 2.2.** Assume that for each \( n = 1, 2, \ldots, \{X_{nk}, k = 1, 2, \ldots, n\} \) be a sequence of independent and identically Bernoulli distributed r-vs. with parameter \( p_n \in (0, 1) \) and \( p_n \to 0, np_n \to \lambda (\lambda > 0) \) as \( n \to \infty \). Then, as \( n \to \infty \),

\[
S_n = \sum_{k=1}^{n} X_{nk} \overset{d}{\to} \text{Poisson}(\lambda).
\]

**Theorem 2.3.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.vs. Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued r-vs. independent on all \( X_j, j = 1, 2, \ldots, n \). Furthermore, assume that \( N_n \sim \text{Geometry}(p) \), \( n \geq 1 \). Then, we have

(i) \( S_{N_n} \sim \text{Exp}(\lambda p) \), when \( X_j \sim \text{Exp}(\lambda), j = 1, 2, \ldots, n \).

(ii) \( S_{N_n} \sim \text{Geometry}(pq) \), when \( X_j \sim \text{Geometry}(q), j = 1, 2, \ldots, n \).

**Theorem 2.4.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.vs. such that \( E(X_1) = 0 \), \( Var(X_1) = 1 \). Moreover, suppose that \( \{N_n, n \geq 1\} \) is a sequence of r.vs. belonging to Poisson law \( \text{Poisson}(\lambda_n) \) and independent of all \( X_n, n \geq 1 \). If

\[
\frac{\lambda_n}{n} \to 1 \quad \text{as} \quad n \to \infty,
\]

then

\[
\frac{S_{N_n}}{\sqrt{n}} \overset{d}{\to} Y^* \quad \text{as} \quad n \to \infty.
\]

**Theorem 2.5.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.vs. such that \( E(X_1) = \mu_1 \), \( E(X^2_1) = \mu_2 \). Suppose that \( \{N_n, n \geq 1\} \) is a sequence of r.vs. from Poisson law \( \text{Poisson}(\lambda_n) \) and independent of all \( X_n, n \geq 1 \). Assume that

\[
\frac{\lambda_n}{n} \to 1 \quad \text{as} \quad n \to \infty
\]

and

\[
\sqrt{n} \left( \frac{\lambda_n}{n} - 1 \right) \to 0 \quad \text{as} \quad n \to \infty.
\]
Then
\[ S_{N_n} - n\mu_1 \over \sqrt{n\mu_2} \xrightarrow{d} X^* \quad \text{as} \quad n \to \infty. \]

**Theorem 2.6.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.v.s. such that \( E(X_1) < \infty, 0 < E(X_1^2) < \infty \). Furthermore, assume that the \( \{N_n, n \geq 1\} \) is a sequence of r.v.s. from Poisson law \( \text{Poisson}(\lambda_n) \) and independent of all \( X_n, n \geq 1 \). If \( \lambda_n \to +\infty \) as \( n \to \infty \), then
\[ S_{N_n} - \lambda_n E(X_1) \over \sqrt{\lambda_n E(X_1^2)} \xrightarrow{d} X^* \quad \text{as} \quad n \to \infty. \]

**Theorem 2.7.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.v.s. such that \( E(X_1) = 0, \text{Var}(X_1) = 1 \). Moreover, suppose that the \( \{N_n, n \geq 1\} \) is a sequence of r.v.s. from Binomial law \( B(n, p) \) and they are independent of all \( X_n, n \geq 1 \). Then
\[ S_{N_n} \over \sqrt{E(N_n)} \xrightarrow{d} X^* \quad \text{as} \quad n \to \infty. \]

**Theorem 2.8.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.v.s. with finite moments. Suppose that \( \{N_n, n \geq 1\} \) is a sequence of r.v.s. from Binomial law \( B(n, p) \) and they are independent of all \( X_n, n \geq 1 \). Then
\[ S_{N_n} - E(S_{N_n}) \over \sqrt{\text{Var}(S_{N_n})} \xrightarrow{d} X^* \quad \text{as} \quad n \to \infty. \]

**Theorem 2.9.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d.r.v.s. such that \( E(X) = 0 \) and \( \text{Var}(X) = 1 \). We assume that \( \{N_n, n \geq 1\} \) is a sequence of random variables of Geometry law \( \text{Geometry}(p_n) \) and they are independent of all \( X_n, n \geq 1 \). If \( p_n \to 0 \) as \( n \to \infty \), then
\[ S_{N_n} \over \sqrt{E(N_n)} \xrightarrow{d} Z \quad \text{as} \quad n \to \infty, \]
where \( Z \) does not belong to \( \mathcal{N}(0,1) \).

**Remark 1.** It is worth pointing out that from Theorem 2.9, we have \( E(N_n) = 1/p_n \to \infty \) as \( n \to \infty \) but the random sum \( S_{N_n} \) does not obey central limit theorem. Therefore, the condition \( E(N_n) \to \infty \), as \( n \to \infty \) is not sufficient for satisfying the central limit theorem for random sum \( S_N \).

**Theorem 2.10.** Let \( \{X_n, n \geq 1\} \) be a sequence of independent standard normal distributed r.v.s. Suppose that \( \{N_n, n \geq 1\} \) is a sequence of positive integer-valued r.v.s. such that the condition (2) is true, and
\[ {E|N_n - E(N_n)| \over E(N_n)} \to 0 \quad \text{as} \quad n \to \infty. \]
Then

\[
\frac{S_{N_n}}{\sqrt{E(N_n)}} \xrightarrow{d} X^* \quad \text{as} \quad n \to \infty.
\]

Remark 2. Based on the fact that \(E|N_n - E(N_n)| \leq \sqrt{\text{Var}(N_n)}\) we can conclude that the condition \(\frac{\text{Var}(N_n)}{E(N_n)} \to 0, \text{ as } n \to \infty\) in Robbins’s results [8] will be stronger than the condition (5).

**Theorem 2.11.** Let \(\{X_n, n \geq 1\}\) be a sequence of i.i.d.r.v.s. such that \(E|X_1| < \infty, E(X_1) = \mu\) and suppose that \(\{N_n, n \geq 1\}\) is a sequence of positive integer-valued r.vs. independent of all \(X_n, n \geq 1\). Assume that the condition (3) true, then

\[
\frac{S_{N_n}}{n} \xrightarrow{p} \mu \quad \text{as} \quad n \to \infty.
\]

**Theorem 2.12.** Let \(\{X_n, n \geq 1\}\) be a sequence of i.i.d.r.v.s., and assume that \(N_n \sim \text{Binomial}(n, p_n), n \geq 1, \text{ satisfying } np_n \to \lambda, \text{ as } n \to \infty.\) Then,

\[
S_{N_n} \xrightarrow{d} S_N \quad \text{as} \quad n \to \infty,
\]

where \(N \sim \text{Poisson}(\lambda)\).

3. Proofs

**Proof of Theorem 2.1.** (i) It is clear that the generating function of r.vs. \(N_n, n \geq 1\) is \(g(t) = e^{\lambda(t-1)}\) and characteristic function of the r.vs. \(X_n, n \geq 1\) is \(\phi(t) = 1 + p(e^{it} - 1)\). Then, the characteristic function of random sum \(S_{N_n}\) is given by

\[
\psi(t) = g(\phi(t)) = e^{\lambda p(e^{it} - 1)}.
\]

Thus \(S_{N_n} \sim \text{Poisson}(\lambda p)\).

(ii) Let us denote \(g(t) = \left[1 + q\varphi(t)\right]^{1/(1-q)}\) the generating function of the r.vs. \(N_n\) and let the characteristic function of the r.vs. \(X_n, n \geq 1\) be \(\varphi(t) = 1 + p(e^{it} - 1)\). Then, the characteristic function of random sum \(S_{N_n}\) can be calculated by

\[
\psi(t) = g(\varphi(t)) = \left[\frac{q}{p+q}e^{it}\right]^{n/(1-q)}.
\]

Therefore \(S_{N_n} \sim \text{Geometry}\left(\frac{q}{p+q}\right)\).

(iii) Let \(g(t) = [1 + q(t - 1)]^n\) and \(\varphi(t) = 1 + p(e^{it} - 1)\) be the generating function and characteristic function of r.vs. \(N_n, n \geq 1\) and \(X_j, j = 1, 2, \ldots,\) respectively. Then, characteristic function of random sum \(S_{N_n}\) will be defined by

\[
\psi(t) = g(\varphi(t)) = [1 + pq(e^{it} - 1)]^n.
\]

By this way, \(S_{N_n} \sim \text{Binomial}(n,pq)\).

\[\square\]
Proof of Theorem 2.2. Let consider \( N_n \sim \text{Poisson}(n) \). According to Theorem 2.1(i) we have \( S_{N_n} \sim \text{Poisson}(np_n) \). Then,
\[
\mathbb{P}(S_{N_n} = k) = \frac{e^{-np_n}(np_n)^k}{k!} \rightarrow e^{-\lambda \lambda^k} \quad \text{as} \quad n \rightarrow \infty,
\]
or \( S_{N_n} \overset{d}{\rightarrow} \text{Poisson}(\lambda) \). Therefore, we must only to show that
\[
\Delta_n(t) = |\varphi_{S_{N_n}}(t) - \varphi_{S_n}(t)| \rightarrow 0.
\]
We contend that
\[
\Delta_n(t) = |e^{np_n(e^{it} - 1)} - [1 - p_n(1 - e^{it})]^n| \leq n|e^{p_n(e^{it} - 1)} - 1 - p_n(e^{it} - 1)|.
\]
Clearly, \( \Re(p_n(e^{it} - 1)) \leq 0 \), applying the inequality \( |e^\alpha - 1 - \alpha| \leq \frac{|\alpha|^2}{2} \) with \( \Re(\alpha) \leq 0 \), we obtain
\[
\Delta_n(t) \leq n|p_n(e^{it} - 1)|^2 = np_n|e^{it} - 1|^2 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
This completes the proof. \( \square \)

Proof of Theorem 2.3. (i) Let us denote the generating function \( g(t) = \frac{pt}{1-(1-p)t} \) and the characteristic function \( \varphi(t) = \frac{\lambda}{\lambda - it} \) of \( N_n \), \( n \geq 1 \) and \( X_j \), \( j = 1, 2, \ldots \) respectively. Then, the characteristic function of random sum \( S_{N_n} \) is given by
\[
\psi(t) = g(\varphi(t)) = \frac{\lambda p}{\lambda p - it}.
\]
We conclude that \( S_{N_n} \sim \text{Exp}(\lambda p) \).

(ii) Let \( g(t) = \frac{pt}{1-(1-p)t} \) be the generating function of \( N_n \), \( n \geq 1 \) and denote \( \varphi(t) = \frac{\lambda}{\lambda - it} \) the characteristic function of \( X_j \), \( j = 1, 2, \ldots \). Then, the characteristic function of random sum \( S_{N_n} \), \( n \geq 1 \) will be given by
\[
\psi(t) = g(\varphi(t)) = \frac{pqe^{it}}{1-(1-pq)e^{it}}.
\]
Hence, \( S_{N_n} \sim \text{Geometry}(pq) \). \( \square \)

Proof of Theorem 2.4. Denote \( g_n(t) = e^{\lambda_n(t-1)} \) the generating function of \( N_n \), \( n \geq 1 \) and \( \varphi \), \( \psi_n \) are characteristic functions of \( X_1 \), \( \frac{S_{N_n}}{\sqrt{n}} \) respectively. Then,
\[
\psi_n(t) = \varphi_{S_{N_n}} \left( \frac{t}{\sqrt{n}} \right) = g_n \left( \varphi \left( \frac{t}{\sqrt{n}} \right) \right) = e^{\lambda_n \left[ \varphi \left( \frac{t}{\sqrt{n}} \right) - 1 \right]},
\]
where \( \varphi \left( \frac{t}{\sqrt{n}} \right) = 1 - \frac{t^2}{2n} + o \left( \frac{1}{n} \right) \).

Hence
\[
\ln \psi_n(t) = \lambda_n \left[ - \frac{t^2}{2n} + o \left( \frac{1}{n} \right) \right] = - \frac{\lambda_n}{n} \cdot \frac{t^2}{2} + o \left( \frac{\lambda_n}{n} \right).
\]
According to assumptions, as \( n \rightarrow \infty \), we have then \( \ln \psi_n(t) \rightarrow -\frac{t^2}{2} \). This finishes the proof. \( \square \)
Proof of Theorem 2.5. Let \( g_n(t) = e^{\lambda_n(t-1)} \) be generating function of \( N_n, \ n \geq 1 \) and denote \( \varphi, \ \psi_n \) are characteristic functions of \( X_1, \ \frac{S_n}{\sqrt{n}} \), respectively. Then,

\[
\psi_n(t) = e^{-it \frac{n\mu_1}{\sqrt{n\mu_2}}} \varphi_{S_n} \left( \frac{t}{\sqrt{n\mu_2}} \right) = e^{-it \frac{n\mu_1}{\sqrt{n\mu_2}}} e^{\lambda_n \left[ \varphi \left( \frac{t}{\sqrt{n\mu_2}} \right) - 1 \right]},
\]

here \( \varphi \left( \frac{t}{\sqrt{n\mu_2}} \right) = 1 + \frac{it\mu_1}{\sqrt{n\mu_2}} - \frac{t^2}{2n} + o \left( \frac{1}{n} \right) \).

From this

\[
\ln \psi_n(t) = -it \frac{n\mu_1}{\sqrt{n\mu_2}} + it \frac{\lambda_n \mu_1}{\sqrt{n\mu_2}} - \frac{\lambda_n}{n} t^2 + o \left( \frac{\lambda_n}{n} \right)
\]

\[
= it \frac{\mu_1}{\sqrt{\mu_2}} n \left( \frac{\lambda_n}{n} - 1 \right) - \frac{\lambda_n}{n} \frac{t^2}{2} + o \left( \frac{\lambda_n}{n} \right).
\]

On account of assumptions, when \( n \to \infty \), we get \( \ln \psi_n(t) \to -\frac{t^2}{2} \). The proof is straightforward.

Proof of Theorem 2.6. Putting \( \mu_1 = E(X_1), \ \mu_2 = E(X_1^2) \) and let \( g_n(t) = e^{\lambda_n(t-1)} \) be a generating function of \( N_n, \ n \geq 1 \) and \( \varphi, \ \psi_n \) be are characteristic functions of \( X_1, \ \frac{S_n}{\sqrt{\lambda_n \mu_2}} \), respectively. Then,

\[
\psi_n(t) = e^{-it \frac{\lambda_n \mu_1}{\sqrt{\lambda_n \mu_2}}} \varphi_{S_n} \left( \frac{t}{\sqrt{\lambda_n \mu_2}} \right) = e^{-it \frac{\lambda_n \mu_1}{\sqrt{\lambda_n \mu_2}}} e^{\lambda_n \left[ \varphi \left( \frac{t}{\sqrt{\lambda_n \mu_2}} \right) - 1 \right]},
\]

where \( \varphi \left( \frac{t}{\sqrt{\lambda_n \mu_2}} \right) = 1 + \frac{it\mu_1}{\sqrt{\lambda_n \mu_2}} - \frac{t^2}{2\lambda_n} + o \left( \frac{1}{\lambda_n} \right) \).

Hence

\[
\ln \psi_n(t) = \lambda_n \left[ - \frac{t^2}{2\lambda_n} + o \left( \frac{1}{\lambda_n} \right) \right] = - \frac{t^2}{2} + o(1).
\]

If \( n \to \infty \), then \( \ln \psi_n(t) \to -\frac{t^2}{2} \). We obtain the proof.

Proof of Theorem 2.7. It is easy to see that the \( N_n, \ n \geq 1 \) have the generating function \( g_n(t) = [1+p(t-1)]^n \) and \( E(N_n) = np \). Let \( \varphi, \ \psi_n \) be are characteristic functions of \( X_1 \) and \( \frac{S_n}{\sqrt{E(N_n)}} \), respectively. Then,

\[
\psi_n(t) = g_n \left( \varphi \left( \frac{t}{\sqrt{mp}} \right) \right) = \left( 1 + p \left[ \varphi \left( \frac{t}{\sqrt{mp}} \right) - 1 \right] \right)^n,
\]

where \( \varphi \left( \frac{t}{\sqrt{mp}} \right) = 1 - \frac{t^2}{2np} + o \left( \frac{1}{n} \right) \).

Therefore,

\[
\psi_n(t) = \left[ 1 - \frac{t^2}{2n} + o \left( \frac{1}{n} \right) \right]^n.
\]

If \( n \to \infty \), then \( \psi_n(t) \to e^{-\frac{t^2}{2}} \). We have the complete proof.
Proof of Theorem 2.8. Let us denote $\mu_1 = E(X_1)$; $\mu_2 = E(X_1^2)$; $\delta = \mu_2 - \mu_1^2/2$. Then,
\[ E(S_{N_n}) = np\mu_1, \quad \text{Var}(S_{N_n}) = np(\mu_2 - \mu_1^2/2) = np\delta. \]
Clearly, $g_n(t) = [1 + p(t - 1)]^n$ is the generating function of $N_n$, $n \geq 1$. We denote $\varphi$ and $\psi_n$ the characteristic functions of $X_1$ and $\frac{S_{N_n} - np\mu_1}{\sqrt{np\delta}}$, respectively.

Then,
\[ \psi_n(t) = e^{-it\mu_1\sqrt{np\delta}} \cdot \varphi(S_{N_n}) \left( \frac{t}{\sqrt{np\delta}} \right) = e^{-it\mu_1\sqrt{np\delta}} \left(1 + p \left[ \varphi \left( \frac{t}{\sqrt{np\delta}} \right) - 1 \right] \right)^n, \]
where $\varphi \left( \frac{t}{\sqrt{np\delta}} \right) = 1 + it\mu_1 \frac{t}{\sqrt{np\delta}} - \frac{t^2 \mu_2}{2np\delta} + o \left( \frac{1}{n} \right)$.

Therefore
\[
\begin{align*}
\psi_n(t) &= e^{-it\mu_1\sqrt{np\delta}} \left[ 1 + it\mu_1 \frac{t}{\sqrt{np\delta}} - \frac{t^2 \mu_2}{2np\delta} - \frac{t^2}{2} + o \left( \frac{1}{n} \right) \right]^n \\
&= e^{-it\mu_1\sqrt{np\delta}} \left[ e^{it\mu_1\sqrt{np\delta}} - \frac{t^2}{2} + o \left( \frac{1}{n} \right) \right]^n \\
&= \left[ 1 - \frac{t^2}{2n} e^{-it\mu_1\sqrt{p/(n\delta)}} + o \left( \frac{1}{n} \right) \right]^n.
\end{align*}
\]

By letting $n \to \infty$, $\psi_n(t) \to e^{-t^2/2}$. We have the proof. \qed

Proof of Theorem 2.9. Let $g_n(t) = \frac{1-t^{a_n}}{1-p_n t}$ be a generating function of the random sum $N_n$, $n \geq 1$ and suppose that $E(N_n) = \frac{1}{p_n}$. We denote $\varphi$ and $\psi_n$ the characteristic functions of $X_1$ and $\sqrt{p_n}S_{N_n}$, respectively.

Then,
\[ \psi_n(t) = g_n(\varphi(t\sqrt{p_n})) = \frac{p_n \varphi(t\sqrt{p_n})}{1 - (1-p_n)\varphi(t\sqrt{p_n})}, \]
where $\varphi(t\sqrt{p_n}) = 1 - p_n t^2/2 + o(p_n)$.

Therefore
\[ \psi_n(t) = \frac{1 - p_n t^2/2 + o(p_n)}{1 + t^2/2 - o(1-p_n)}. \]

From the assumptions, by letting $n \to \infty$, then $\psi_n(t) \to e^{-t^2/2} \neq e^{-t^2/2}$. We get the complete proof. \qed

Proof of Theorem 2.10. Suppose that $X_j$, $j = 1, 2, \ldots$ having the characteristic function $\varphi(t) = e^{-t^2/2}$. Let us put $g_n(t) = E(t^{N_n})$, $p_k = \mathbb{P}(N_n = k)$, $a_n = E(N_n)$. Moreover, suppose that $\psi_n(t)$ is a characteristic function of $\frac{S_{N_n}}{\sqrt{a_n}}$. Then,
\[ \psi_n(t) = g_n \left( \varphi \left( \frac{t}{\sqrt{a_n}} \right) \right) = \sum_{k=0}^{\infty} p_k e^{-t^2k/2a_n}. \]
Therefore
\[ \left| \psi_n(t) - e^{-t^2/2} \right| = \sum_{k=0}^{\infty} p_k \left( e^{-\frac{t^2}{2a_n}} - e^{-\frac{t^2}{2}} \right) \leq \sum_{k=0}^{\infty} p_k \left| e^{-\frac{t^2}{2a_n}} - e^{-\frac{t^2}{2}} \right|. \]

In another way, using the Lagrange's Theorem for the function \( h(x) = (e^{-t^2/2})^x \) continuing on \([k a_n, 1]\) or \([1, k a_n]\), we have
\[ \left| e^{-\frac{t^2}{2k a_n}} - e^{-\frac{t^2}{2}} \right| = \left| \frac{k}{a_n} - 1 \right| \left| h'(c) \right| \text{ (for any } c) \]
\[ = \frac{t^2}{2} \left| \frac{k}{a_n} - 1 \right| \left| h'(c) \right| \leq \frac{t^2}{2} \left| \frac{k}{a_n} - 1 \right|. \]
(because of \( c \geq 0, \ h(x) \text{ is decreasing})

Then
\[ \left| \psi_n(t) - e^{-t^2/2} \right| \leq \sum_{k=0}^{\infty} p_k \left| \frac{k}{a_n} - 1 \right| \frac{t^2}{2} = \frac{t^2}{2} \left| \frac{N_n - a_n}{a_n} \right|. \]

According to assumptions, by letting \( n \to \infty \), then \( \psi_n(t) \to e^{-t^2/2} \). We have the proof.

**Proof of Theorem 2.11.** According to the Weak Law of Large Numbers, we have \( \frac{S_n}{n} \overset{p}{\to} \mu \). Because of assumptions, using the considerations on relationships among the conditions (3) and (4), it follows \( N_n \overset{p}{\to} \infty \). Since, for \( \epsilon > 0, 3n_0, \forall n > n_0 : \mathbb{P}(|\frac{S_n}{n} - \mu| > \epsilon) < \epsilon, \)
\[ \mathbb{P} \left( \left| \frac{S_n}{N_n} - \mu \right| > \epsilon \right) = \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) \mathbb{P} \left( \left| \frac{S_k}{k} - \mu \right| > \epsilon \right) = \sum_{k=1}^{n_0} + \sum_{k=n_0+1}^{\infty} \mathbb{P}(N_n \leq n_0) + \epsilon, \]
it is easy to derive \( \frac{S_n}{N_n} \overset{p}{\to} \mu \). Then,
\[ \frac{S_n}{n} = \frac{S_n}{N_n} \cdot \frac{N_n}{n} \overset{p}{\to} \mu. \]
We obtain the proof.

**Proof of Theorem 2.12.** Let \( g_n(t) = [1 + p_n(t - 1)]^n \) be a generating function of \( N_n \) and suppose that \( \varphi \) and \( \psi_n \) are characteristic functions of \( X_1 \) and \( S_{N_n} \), respectively. Then
\[ \psi_n = g_n(\varphi(t)) = (1 + p_n[\varphi(t) - 1])^n = (1 + \frac{n p_n}{n} [\varphi(t) - 1])^n. \]
By putting \( n \to \infty \), then \( \psi_n(t) \to e^{A[\varphi(t)-1]} \). The proof is finished.

**Acknowledgements.** The authors would like to express their gratitude to referees for several constructive suggestions.
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