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CHOONKIL PARK AND JONG SU AN

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CHOONKIL PARK AND JONG SU AN

ABSTRACT. Using the Hyers–Ulam–Rassias stability method, we investigate isomorphisms in quasi-Banach algebras and derivations on quasi-Banach algebras associated with the Cauchy–Jensen functional equation

$$2f\left(\frac{x+y}{2} + z\right) = f(x) + f(y) + 2f(z),$$

which was introduced and investigated in [2, 17]. The concept of Hyers–Ulam–Rassias stability originated from the Th. M. Rassias' stability theorem that appeared in the paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300. Furthermore, isometries and isometric isomorphisms in quasi-Banach algebras are studied.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let (G_2, \diamond, d) be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

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Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \rightarrow Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : X \rightarrow Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th. M. Rassias [19] introduced the following inequality: Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Th. M. Rassias [19] showed that there exists a unique \mathbb{R} -linear mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in X$. The above inequality has provided a lot of influence in the development of what is now known as *Hyers–Ulam–Rassias stability* of functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [13]–[16], [20], [22]–[25]).

Gilányi [8] showed that if f satisfies the functional inequality

$$(1.1) \quad \|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\|,$$

then f satisfies the Jordan–von Neumann functional inequality

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [27]. Fechner [7] and Gilányi [9] proved the Hyers–Ulam–Rassias stability of the functional inequality (1.1). Park, Cho, and Han [18] proved the Hyers–Ulam–Rassias stability of functional inequalities associated with Jordan–von Neumann type additive functional equations.

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 ([5], [28]). Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
- (3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X .

A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p -norm ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p -Banach space.

Given a p -norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki–Rolewicz theorem [28] (see also [5]), each quasi-norm is equivalent to some p -norm. Since it is much easier to work with p -norms than quasi-norms, henceforth we restrict our attention mainly to p -norms.

Definition 1.2 ([1]). Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $K > 0$ such that $\|xy\| \leq K\|x\| \cdot \|y\|$ for all $x, y \in A$.

A *quasi-Banach algebra* is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a p -norm then the quasi-Banach algebra is called a p -Banach algebra.

Throughout this paper, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and unit e and that B is a p -Banach algebra with p -norm $\|\cdot\|_B$ and unit e' .

This paper is organized as follows: In Section 2, we investigate isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

In Section 3, we investigate derivations on quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

In Section 4, we investigate isometries and isometric isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen additive mapping.

2. Isomorphisms in quasi-Banach algebras

Definition 2.1. A \mathbb{C} -linear mapping $H : A \rightarrow B$ is called a *homomorphism in quasi-Banach algebras* if $H(xy) = H(x)H(y)$ for all $x, y \in A$. If, in addition, the mapping $H : A \rightarrow B$ is bijective, then the mapping $H : A \rightarrow B$ is called an *isomorphism in quasi-Banach algebras*.

In this section, we investigate isomorphisms in quasi-Banach algebras associated with the Cauchy–Jensen functional equation.

Theorem 2.2. Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow B$ a bijective mapping such that

$$(2.1) \quad \|\mu f(x) + f(y) + 2f(z)\|_B \leq \|2f(\frac{\mu x + y}{2} + z)\|_B,$$

$$(2.2) \quad \|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^{2r} + \|y\|_A^{2r})$$

for $\mu = 1, i$ and all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $f : A \rightarrow B$ is an isomorphism in quasi-Banach algebras.

Proof. Let $\mu = 1$ in (2.1). By Proposition 2.3 of [18], the mapping $f : A \rightarrow B$ is Cauchy additive. By Theorem of [19], the mapping $f : A \rightarrow B$ is \mathbb{R} -linear.

Letting $\mu = i$, $z = 0$ and $y = -ix$ in (2.1), we get

$$if(x) - f(ix) = if(x) + f(-ix) = 0$$

for all $x \in A$. So $f(ix) = if(x)$ for all $x \in A$. For each $\lambda \in \mathbb{C}$, $\lambda = a + ib$ ($a, b \in \mathbb{R}$). Hence

$$f(\lambda x) = f(ax + ibx) = af(x) + bf(ix) = af(x) + ibf(x) = \lambda f(x)$$

for all $x \in A$. Thus $f : A \rightarrow B$ is \mathbb{C} -linear.

(i) Assume that $r < 1$. By (2.2),

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\leq \lim_{n \rightarrow \infty} \frac{4^{nr}}{4^n} \theta (\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \rightarrow B$ satisfies

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

Therefore, the bijective mapping $f : A \rightarrow B$ is an isomorphism in quasi-Banach algebras, as desired. \square

Theorem 2.3. Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow B$ a bijective mapping satisfying (2.1) such that

$$(2.3) \quad \|f(xy) - f(x)f(y)\|_B \leq \theta \cdot \|w\|_A^r \cdot \|x\|_A^r$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $f : A \rightarrow B$ is an isomorphism in quasi-Banach algebras.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \rightarrow B$ is \mathbb{C} -linear.

(i) Assume that $r < 1$. By (2.3),

$$\begin{aligned} \|f(xy) - f(x)f(y)\|_B &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x)f(2^n y)\|_B \\ &\leq \lim_{n \rightarrow \infty} \frac{4^{nr}}{4^n} \theta \cdot \|w\|_A^r \cdot \|x\|_A^r = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \rightarrow B$ satisfies

$$f(xy) = f(x)f(y)$$

for all $x, y \in A$.

Therefore, the bijective mapping $f : A \rightarrow B$ is an isomorphism in quasi-Banach algebras, as desired. \square

3. Derivations on quasi-Banach algebras

Definition 3.1. A \mathbb{C} -linear mapping $\delta : A \rightarrow A$ is called a *derivation* if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We investigate derivations on quasi-Banach algebras associated with the Cauchy–Jensen functional equation.

Theorem 3.2. Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow A$ a mapping such that

$$(3.1) \quad \|\mu f(x) + f(y) + 2f(z)\|_A \leq \|2f(\frac{\mu x + y}{2} + z)\|_A,$$

$$(3.2) \quad \|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^{2r} + \|y\|_A^{2r})$$

for $\mu = 1, i$ and all $x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \rightarrow A$ is a derivation on A .

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \rightarrow A$ is \mathbb{C} -linear.

(i) Assume that $r < 1$. By (3.2),

$$\begin{aligned} \|f(xy) - f(x)y - xf(y)\|_A &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x) \cdot 2^n y - 2^n x f(2^n y)\|_A \\ &\leq \lim_{n \rightarrow \infty} \frac{4^{nr}}{4^n} \theta(\|x\|_A^{2r} + \|y\|_A^{2r}) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \rightarrow A$ satisfies

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in A$.

Therefore, the mapping $f : A \rightarrow A$ is a derivation on A , as desired. \square

Theorem 3.3. Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow A$ a mapping satisfying (3.1) such that

$$(3.3) \quad \|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \rightarrow A$ is a derivation on A .

Proof. The proof is similar to the proofs of Theorems 2.2 and 3.2. \square

4. Isometries and isometric isomorphisms in quasi-Banach algebras

Surjective isometries between normed vector spaces have been investigated by several authors ([4], [6], [11], [12], [21], [26]).

Definition 4.1. A mapping $I : A \rightarrow B$ is called an *isometry in quasi-Banach algebras* if

$$\|I(x) - I(y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

We investigate isometries in quasi-Banach algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.2. Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow B$ a mapping satisfying (2.1) such that

$$(4.1) \quad |\|f(x)\|_B - \|x\|_A| \leq \theta \|x\|_A^r$$

for all $x \in A$. Then the mapping $f : A \rightarrow B$ is an isometry in quasi-Banach algebras.

Proof. By the same reasoning as in the proof of Theorem 2.2, the mapping $f : A \rightarrow B$ is additive.

(i) Assume that $r < 1$. It follows from (4.1) that

$$|\|f(x)\|_B - \|x\|_A| = \lim_{n \rightarrow \infty} \frac{1}{2^n} |\|f(2^n x)\|_B - \|2^n x\|_A| \leq \lim_{n \rightarrow \infty} \frac{2^{nr}}{2^n} \theta \|x\|_A^r = 0$$

for all $x \in A$. So $\|f(x)\|_B = \|x\|_A$ for all $x \in A$. Since $f : A \rightarrow B$ is additive,

$$\|f(x) - f(y)\|_B = \|f(x - y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

(ii) Assume that $r > 1$. By a similar method to the proof of the case (i), one can prove that the mapping $f : A \rightarrow B$ satisfies

$$\|f(x) - f(y)\|_B = \|x - y\|_A$$

for all $x, y \in A$.

Therefore, the mapping $f : A \rightarrow B$ is an isometry in quasi-Banach algebras. \square

Definition 4.3. A quasi-Banach algebra isomorphism $H : A \rightarrow B$ is called an *isometric isomorphism in quasi-Banach algebras* if H is an isometry in quasi-Banach algebras.

We investigate isometric isomorphisms in quasi-Banach algebras associated with the Cauchy-Jensen functional equation.

Theorem 4.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and $f : A \rightarrow B$ a bijective mapping satisfying (2.1), (2.2) and (4.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \rightarrow B$ is an isometric isomorphism in quasi-Banach algebras.*

Proof. The proof is similar to the proofs of Theorems 2.2 and 4.2. □

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CHOONKIL PARK
DEPARTMENT OF MATHEMATICS
HANYANG UNIVERSITY
SEOUL 133-791, KOREA
E-mail address: baak@hanyang.ac.kr

JONG SU AN
DEPARTMENT OF MATHEMATICS EDUCATION
PUSAN NATIONAL UNIVERSITY
PUSAN 609-735, KOREA
E-mail address: jsan63@pusan.ac.kr