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ABSTRACT. In this paper, we modify L. Cădariu and V. Radu’s result for the stability of the monomial functional equation

\[ \sum_{i=0}^{n} n! C_i (-1)^{n-i} f(ix + y) - n! f(x) = 0 \]

in the sense of Th. M. Rassias. Also, we investigate the superstability of the monomial functional equation.

1. Introduction

Throughout this paper, let \( X \) be a vector space and \( Y \) a Banach space. Let \( n \) be a positive integer. For a given mapping \( f : X \to Y \), define a mapping \( D_n f : X \times X \to Y \) by

\[ D_n f(x, y) := \sum_{i=0}^{n} n! C_i (-1)^{n-i} f(ix + y) - n! f(x) \]

for all \( x, y \in X \), where \( n! C_i = \frac{n!}{i!(n-i)!} \). A mapping \( f : X \to Y \) is called a monomial function of degree \( n \) if \( f \) satisfies the monomial functional equation \( D_n f(x, y) = 0 \). The function \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) := ax^n \) is a particular solution of the functional equation \( D_n f = 0 \). In particular, a mapping \( f : X \times X \to Y \) is called an additive (quadratic, cubic, quartic, respectively) mapping if \( f \) satisfies the functional equation \( D_1 f = 0 \) (\( D_2 f = 0 \), \( D_3 f = 0 \), \( D_4 f = 0 \), respectively).

In 1940, S. M. Ulam [27] raised a question concerning the stability of homomorphisms: Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exists a \( \delta > 0 \) such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality

\[ d(h(xy), h(x)h(y)) < \delta \]

for all \( x, y \in G_1 \) then there is a homomorphism \( H : G_1 \to G_2 \) with

\[ d(h(x), H(x)) < \varepsilon \]

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for all $x \in G_1$?

In 1941, D. H. Hyers [7] proved the stability theorem for additive functional equation $D_1 f = 0$ under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Th. M. Rassias [20] provided an extension of D. H. Hyers’s Theorem by proving the generalized Hyers-Ulam stability for the linear mapping subject to the unbounded Cauchy difference that he introduced in [20]. Th. M. Rassias’s Theorem provided a lot of influence for the rapid development of stability theory of functional equations during the last three decades. This generalized concept of stability is known today with the term Hyers-Ulam-Rassias stability of the linear mapping or of functional equations. Further generalizations of the Hyers-Ulam-Rassias stability concept have been investigated by a number of mathematicians worldwide (cf. [5, 6, 8, 9, 11, 12, 14, 17-19, 21-25]). In 1983, the Hyers-Ulam-Rassias stability theorem for the quadratic functional equation $D_2 f = 0$ was proved by F. Skof [26] and a number of other mathematicians (cf. [2, 3, 4, 10, 13]). The Hyers-Ulam-Rassias stability Theorem for the functional equation $D_3 f = 0$ and $D_4 f = 0$ was proved by J. Rassias [15, 16].

In 2007, L. Cădariu and V. Radu [1] proved the stability of the monomial functional equation $D_n f = 0$.

In this paper, we modify L. Cădariu and V. Radu’s result for the stability of the monomial functional equation $D_n f = 0$ in the sense of Th. M. Rassias and the superstability of the monomial functional equation $D_n f = 0$.

2. The stability of the monomial functional equation

Since the equalities

$$(1 - x^2)^n = \sum_{i=0}^{n} nC_i (-1)^i x^{2i},$$

$$(1 - x)^n(x + 1)^n = \left(\sum_{k=0}^{n} nC_k (-1)^k x^k\right)\left(\sum_{j=0}^{n} nC_j x^j\right) = \sum_{i=0}^{n} \sum_{l=0}^{2i} nC_i \cdot nC_{2i-l} (-1)^l x^{2i}$$

hold for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the equality

$$nC_i (-1)^i = \sum_{l=0}^{2i} nC_i \cdot nC_{2i-l} (-1)^l$$

holds for all $n \in \mathbb{N}$.

**Lemma 1.** Let $f : X \rightarrow Y$ be a mapping satisfying the functional equation

$$D_n f(x, y) := \sum_{i=0}^{n} nC_i (-1)^{n-i} f(ix + y) - n! f(x)$$
for all $x, y \in X$. Then equality

$$f(2x) = 2^n f(x)$$

holds for all $x \in X$.

Proof. Using the equalities

$$nC_i(-1)^i = \sum_{l=0}^{2i} nC_l \cdot nC_{2i-l}(-1)^l$$
and

$$\sum_{i=0}^{n} nC_i(-1)^i = 0,$$

the equality

$$n!(f(2x) - 2^n f(x)) = D_n f(2x, (-k)x) - \sum_{j=0}^{n} nC_j D_n f(x, (j-k)x) = 0$$
holds for all $x \in X$ and $k \in \mathbb{N}$ as we desired. \hfill $\Box$

Now, we prove the stability of the monomial functional equation in the sense of Th. M. Rassias.

**Theorem 2.** Let $p$ be a real number with $0 \leq p < n$ and $X$ a normed space. Let $f : X \to Y$ be a mapping such that

$$\|D_n f(x, y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then there exists a unique monomial function of degree $n$ $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\varepsilon}{n!} \cdot \frac{2^n}{2^n} \inf_{k \in \mathbb{N}} \frac{1}{2^n + 2^n + \sum_{j=0}^{n} nC_j |j-k|^p}$$

holds for all $x \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x) := \lim_{s \to \infty} \frac{f(2^s x)}{2^{ns}}$$

for all $x \in X$.

Proof. By (1), we get

$$\|n!(f(2x) - 2^n f(x))\| = \|D_n f(2x, (-k)x) - \sum_{j=0}^{n} nC_j D_n f(x, (j-k)x)\|$$

$$\leq \varepsilon(\|2x\|^p + \|kx\|^p + \sum_{j=0}^{n} nC_j \|x\|^p + \|(j-k)x\|^p)$$

$$= (2^p + k^p + 2^n + \sum_{j=0}^{n} nC_j |j-k|^p)\|x\|^p$$

for all $x \in X$ and $k \in \mathbb{N}$. Hence

$$\|f(x) - \frac{f(2x)}{2^n}\| \leq \frac{\varepsilon}{n! \cdot 2^n} \inf_{k \in \mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} nC_j |j-k|^p)\|x\|^p$$
and
(4)
\[ \| f(x) - \frac{f(2^m x)}{2^{nm}} \| \leq \sum_{s=0}^{m-1} \left\| \frac{f(2^s x)}{2^{sn}} - \frac{f(2^{s+1} x)}{2^{(s+1)n}} \right\| \]
\[ \leq \frac{\varepsilon}{n!} \cdot 2^n \inf_{k \in \mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j|^p) \sum_{s=0}^{m-1} \frac{2^{sp}}{2^{sn}} \|x\|^p \]
for all \( x \in X \). The sequence \( \left\{ \frac{f(2^s x)}{2^{sn}} \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{f(2^s x)}{2^{sn}} \right\} \) converges for all \( x \in X \). Define \( F : X \to Y \) by
\[ F(x) := \lim_{s \to \infty} \frac{f(2^s x)}{2^{sn}} \]
for all \( x \in X \). By (1) and the definition of \( F \), we obtain
\[ D_n F(x, y) = \lim_{s \to \infty} \frac{D_n f(2^s x, 2^s y)}{2^{ns}} = 0 \]
for all \( x, y \in X \). Taking \( m \to \infty \) in (4), we can obtain the inequality (2) for all \( x \in X \).

Now, let \( F' : X \times X \to Y \) be another monomial function satisfying (2). By Lemma 1, we have
\[ \| F(x) - F'(x) \| \leq \left\| \frac{1}{2^{ns}} (F - f)(2^s x) \right\| + \left\| \frac{1}{2^{ns}} (f - F')(2^s x) \right\| \]
\[ \leq \frac{2^{np}}{2^{ns}} n! \inf_{k \in \mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j|^p) \frac{\varepsilon}{2^p} \|x\|^p \]
for all \( x, y \in X \) and \( s \in \mathbb{N} \). As \( s \to \infty \), we may conclude that \( F(x) = F'(x) \) for all \( x \) as desired. \( \square \)

**Theorem 3.** Let \( p \) be a real number with \( p > n \) and \( X \) a normed space. Let \( f : X \to Y \) be a mapping satisfying (1) for all \( x, y \in X \). Then there exists a unique monomial function of degree \( n \) \( F : X \to Y \) such that
\[ \| f(x) - F(x) \| \leq \frac{1}{n!} \inf_{k \in \mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j|^p) \frac{\varepsilon}{2^p} \|x\|^p \]
holds for all \( x \in X \). The mapping \( F : X \times X \to Y \) is given by
\[ F(x) := \lim_{s \to \infty} 2^{ns} f(2^{-s} x) \]
for all \( x \in X \).

**Proof.** By (3), we get
\[ \| f(x) - 2^n f\left(\frac{x}{2}\right) \| \leq \frac{\varepsilon}{n!} \cdot 2^n \inf_{k \in \mathbb{N}} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j|^p) \|x\|^p \]
for all \( x \in X \) and \( k \in \mathbb{N} \). The rest of the proof is similar with the proof of Theorem 2. \( \square \)

3. The superstability of the functional equation \( D_n f = 0 \)

**Lemma 4.** Let \( p \) be a real number with \( p < 0 \) and \( X \) a normed space. Let \( f : X \to Y \) be a mapping satisfying (1) for all \( x, y \in X \setminus \{0\} \). Then there exists a unique monomial function of degree \( n \) \( F : X \to Y \) such that

\[
\| f(x) - F(x) \| \leq \frac{2^p + 2^n}{n!(2^n - 2^p)} \varepsilon \| x \|^p
\]

holds for all \( x \in X \setminus \{0\} \).

**Proof.** As in the proof of Theorem 2, the inequality

\[
\| f(x) - \frac{f(2^m x)}{2^m} \| \leq \sum_{s=0}^{m-1} \| \frac{f(2^s x)}{2^s} - \frac{f(2^{s+1} x)}{2^{s+1}} \|
\]

\[
\leq \frac{\varepsilon}{n!} \inf_{k \geq n+1} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j - k|^p) \sum_{s=0}^{m-1} \frac{2^p}{2^m} \| x \|^p
\]

holds for all \( x \in X \setminus \{0\} \). Since \( p < 0 \), we get

\[
\inf_{k \geq n+1} (2^p + k^p + 2^n + \sum_{j=0}^{n} C_j |j - k|^p) = (2^p + 2^n)
\]

for all \( x \in X \setminus \{0\} \). The rest of the proof is the same to the proof of Theorem 2. \( \square \)

Now, we prove the superstability of the monomial functional equation.

**Theorem 5.** Let \( p \) be a real number with \( p < 0 \) and \( X \) a normed space. Let \( f : X \to Y \) be a mapping satisfying (1) for all \( x, y \in X \setminus \{0\} \). Then \( f \) is a monomial function of degree \( n \).

**Proof.** Let \( F \) be the monomial function of degree \( n \) satisfying (5). From (1), the inequality

\[
\| f(x) - F(x) \|
\]

\[
\leq \frac{1}{n} \| D_n(f - F)((k + 1)x, -kx) + (-1)^n(F - f)(-kx)
\]

\[
+ \sum_{i=2}^{n} nC_i(-1)^{n-i}(F - f)(i(k + 1)x - kx) - n!(F - f)((k + 1)x)\|
\]

\[
\leq \frac{1}{n} \left[ \frac{2^p + 2^n}{n!(2^n - 2^p)} (k^p + \sum_{i=1}^{n-1} nC_{i+1} (ik + i + 1)^p + n!(k + 1)^p) \right]
\]

\[
+ (k + 1)^p + k^p \| x \|
\]

holds for all \( x \in X \setminus \{0\} \). The rest of the proof is the same to the proof of Theorem 2. \( \square \)
holds for all \( x \in X \setminus \{0\} \) and \( k \in \mathbb{N} \). Since \( \lim_{k \to \infty} (k^p + \sum_{i=1}^{n-1} C_{i+1}^i (ik + i + 1)^p + n!(k+1)^p) = 0 \) and \( \lim_{k \to \infty} (k^p + (k+1)^p) = 0 \) for \( p < 0 \), we get

\[
f(x) = F(x)
\]

for all \( x \in X \setminus \{0\} \). Since \( \lim_{k \to \infty} k^p = 0 \) and the inequality

\[
\|f(0) - F(0)\| \leq \frac{1}{n} \|D_n(f - F)(kx, -kx) + (-1)^n(F - f)(-kx)
\]

\[
+ \sum_{i=2}^{n} nC_i(-1)^{n-i}(F - f)((i-1)kx) - n!(F - f)(kx)\| \leq \frac{1}{n} \left[ 2 + \frac{2^n + 2^n}{n(2^n - 2^n)} \left( 1 + \sum_{i=1}^{n-1} nC_{i+1}i^p + n! \right) k^p \|x\|^p \right]
\]

holds for any \( x \in X \setminus \{0\} \) and \( k \in \mathbb{N} \), we get

\[
f(0) = F(0).
\]

\[\square\]

References


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