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WHICH K3 SURFACES WITH PICARD NUMBER 19 COVER AN ENRIQUES SURFACE

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Abstract. We determine which K3 surface with Picard number 19 is a K3 cover of an Enriques surface.

1. Introduction

Let X be a K3 surface with Picard number, $\rho(X)$, 19 over the field $\mathbb{C}$. T_X can be denoted by the following intersection matrix

$$T_X = \begin{pmatrix} 2a & d & e \\ d & 2b & f \\ e & f & 2c \end{pmatrix} \tag{1}$$

with respect to a basis $\{x, y, z\}$. Since the transcendental lattice T_X of X has signature $(2, 1)$, without loss of generality, we may assume that $z^2 = 2c < 0$. Let U and E_8 denote the even unimodular lattices of signature $(1, 1)$ and $(0, 8)$ respectively. Keum showed that every algebraic Kummer surface is the K3 cover of some Enriques surface in [3] with the following criterion.

Theorem 1.1 (Keum, [3]). (*Criterion for a K3 surface to cover an Enriques surface*) *Let X be an algebraic K3 surface. Assume that $l(T_X) + 2 \le \rho(X)$, where $l(T_X)$ is the length of T_X (This is true if $\rho(X) \ge 12$). Then, the following are equivalent.*

(i) *X admits a fixed-point-free involution.*

(ii) *There exists a primitive embedding of T_X into $\Lambda^- = U \oplus U(2) \oplus E_8(2)$ such that the orthogonal complement of T_X in Λ^- contains no vectors of self-intersection -2.*

Following the work of Keum, Ali Sinan Sertöz determined the necessary and sufficient conditions for a singular K3 surface ($\rho(X) = 20$) to cover an Enriques surface, [5]. He used the following lemma to show that a given lattice embedding is a primitive.

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Lemma 1.2 ([5]). *A lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of basis is* 1.

In this paper we will use these for the case $\rho(X) = 19$. For the definitions and basic facts about K3 surfaces we refer to [1].

Our purpose is to show following theorems:

Theorem 1.3. *If X is a K3 surface with a transcendental lattice given as in* (1)*, then the K3 surface satisfying one of the following conditions is a K3 cover of an Enriques surface.*

1. *$a, b, c,$ and def are even.*
2. *a is odd; b and c are even* $\begin{cases} \text{(i) } def \text{ is odd.} \\ \text{(ii) } f \text{ is even and } d \text{ is odd} \\ \quad \text{or } f \text{ is even and } e \text{ is odd.} \end{cases}$
3. *b is odd; a and c are even* $\begin{cases} \text{(i) } e \text{ is even and } d \text{ is odd} \\ \quad \text{or } e \text{ is even and } f \text{ is odd.} \\ \text{(ii) } def \text{ is odd.} \end{cases}$
4. *c is odd; a and b are even* $\begin{cases} \text{(i) } d \text{ is even and } e \text{ or } f \text{ is odd.} \\ \text{(ii) } def \text{ is odd.} \end{cases}$
5. *Only a and f are even.*
6. *Only b and e are even.*
7. *Only c and d are even.*

Theorem 1.4. *If X is a K3 surface with a transcendental lattice given as in* (1)*, then the K3 surface satisfying one of the following conditions is* ***not*** *a K3 cover of any Enriques surface.*

1. *$a, b,$ and c are even; def is odd.*
2. *af is odd; $b, c,$ and de are even.*
3. *be is odd; $a, c,$ and df are even.*
4. *cd is odd; $a, b,$ and ef are even.*
5. *a is even and bc is odd* $\begin{cases} \text{(i) } f \text{ is odd.} \\ \text{(ii) } f \text{ is even and } d + e \text{ is odd.} \end{cases}$
6. *b is even and ac is odd* $\begin{cases} \text{(i) } e \text{ is odd.} \\ \text{(ii) } e \text{ is even and } d + f \text{ are odd.} \end{cases}$
7. *c is even and ab is odd* $\begin{cases} \text{(i) } d \text{ is odd.} \\ \text{(ii) } d \text{ is even and } e + f \text{ is odd.} \end{cases}$
8. *abc is odd; d, e, or f is odd.*

Remark 1.5. The remaining cases are as follows:

1. Only a is odd.
2. Only b is odd.
3. Only c is odd.
4. Only a and b are odd.
5. Only b and c are odd.
6. Only a and c are odd.

7. Only a, b, and c are odd.

However, since we will show that these are all equivalent, it is sufficient to consider only one of these.

Before we proceed to the proof of the theorems we will also need the following lemma. We recall that for $A \in SL_3(\mathbb{Z})$, $T'_X = AT_XA^{\text{tr}} = \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$ is $\mathbb{Z}$-equivalent to T_X.

Lemma 1.6. *Let T_X be a lattice given as in* (1) *with $c < 0$. Then, T_X is $\mathbb{Z}$-equivalent to T'_X with $b', c' < 0$ and $a' = a$.*

Proof. Assume that $b \geq 0$ (If $b < 0$, there is nothing to prove). Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$, where $\alpha \in \mathbb{Z}$. Since $A \in SL_3(\mathbb{Z})$, the matrix AT_XA^{tr} is $\mathbb{Z}$-equivalent to T_X. Now $T'_X = AT_XA^{tr} = \begin{pmatrix} 2a' & d' & e' \\ d' & 2b' & f' \\ e' & f' & 2c' \end{pmatrix}$, where $a' = a$, $b' = b + \alpha^2 c + \alpha f$, $c' = c$, $d' = d + \alpha e$, $e' = e$, $f' = 2\alpha c + f$.
<u>Case 1.</u> $f \neq 0$.
Let $\alpha = -nf$, where $n \in \mathbb{Z}^+$. Then $b' = b + (-nf)^2 c + (-n)f^2 < 0$ for some n.
<u>Case 2.</u> $f = 0$.
We can take $\alpha = n \in \mathbb{Z}$ such that $b' < 0$, where $b' = b + n^2 c$.
<u>Case 3.</u> $b = 0$.
If $f \leq 0$, let $\alpha = 1$. Then $b' = c + f < 0$. If $f > 0$, let $\alpha = -1$. Then $b' = c - f < 0$. ☐

From now on, we assume that b and c in T_X of (1) are negative.

Theorem 1.7 (Nikulin, [4])**.** *A primitive embedding of an even non-degenerate lattice L of signature (s_+, s_-) into an even unimodular lattice M of signature (t_+, t_-) exists provided that*

$$s_+ \leq t_+, s_- \leq t_-, \text{ and } l(L) + 1 \leq \operatorname{rank}(M) - \operatorname{rank}(L),$$

where $l(L)$ is the length of L. Furthermore, if the three inequalities are all strict, then the primitive embedding is unique.

The following corollary will be used later.

Corollary 1.8. *There is a primitive embedding of $\langle -2m \rangle$ into E_8 for any positive integer m.*

2. Proof of Theorem 1.3

Let $\{x, y, z\}$ be a basis of the transcendental lattice T_X and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard bases of U and $U(2)$, respectively. We prove Theorem 1.3 by showing the existence of primitive embedding of T_X into Λ^- such that the orthogonal complement of $\text{Im}\phi$ contains no (-2)-vectors.

1-i) a, b, c, and e are even.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined as

$$\phi(x) = du_2 + v_1 + \frac{a}{2}v_2,$$
$$\phi(y) = u_1 + bu_2,$$
$$\phi(z) = fu_2 + \frac{e}{2}v_2 + w,$$

where by Corollary 1.8 we can choose a primitive element w of $E_8(2)$ with $w^2 = 2c$, $c < 0$. Then ϕ is an embedding and by Lemma 1.2, ϕ is a primitive embedding. Assume that $s = x_1u_1 + x_2u_2 + x_3v_1 + x_4v_2 + w'$ is an element of orthogonal complement of the Imϕ. Then $s \cdot \phi(y) = 0$ induces $bx_1 + x_2 = 0$. Since b is even, x_2 is even. Thus $s \cdot s = 2x_1x_2 + 4x_3x_4 + w'^2 \equiv 0 \pmod 4$ and hence cannot be -2.

1-ii) a, b, c, and f are even.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined as

$$\phi(x) = u_1 + au_2,$$
$$\phi(y) = du_2 + v_1 + \frac{b}{2}v_2,$$
$$\phi(z) = eu_2 + \frac{f}{2}v_2 + w,$$

where by Corollary 1.8 we can choose a primitive element w of $E_8(2)$ with $w^2 = 2c$, $c < 0$. This is an embedding and by Lemma 1.2, it is primitive. Assume that $s = x_1u_1 + x_2u_2 + x_3v_1 + x_4v_2 + w'$ is an element of orthogonal complement of the Imϕ. Then $s \cdot \phi(x) = 0$ induces $ax_1 + x_2 = 0$. Since a is even, x_2 is even. Thus $s \cdot s \equiv 0 \pmod 4$ and hence cannot be -2.

1-iii) a, b, c, and d are even (ef is odd, otherwise 1-i) or 1-ii)).
We use the base change by $\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, where $a' = a + b + d$, $b' = b$, $c' = c$, $d' = 2b + d$, $e' = e + f$, and $f' = f$. Then a', b', c', and e' are even. Thus this case is reduced to Case 1-i).

2-i) Only b and c are even.
We use the base change by $\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, where $a' = a + b + d$, $b' = b$, $c' = c$, $d' = 2b + d$, $e' = e + f$, and $f' = f$. Then a', b', c', and e' are even. Thus this case is reduced to Case 1.

2-ii) b, c, and f are even, a is odd, and either d or e is odd.
If d is odd, then we use the base change by $\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, where $a' = a + b + d$, $b' = b$, $c' = c$, $d' = 2b + d$, $e' = e + f$, and $f' = f$. Then a', b', c', and f' are even. Thus this case is reduced to Case 1.
If e is odd, then we use the base change by $\begin{pmatrix} 1&0&1\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, where $a' = a + c + e$, $b' = b$, $c' = c$, $d' = d + f$, $e' = 2c + e$, and $f' = f$. Then a', b', c', and f' are even. Thus this case is reduced to Case 1.

3-i) a, c, and e are even, b is odd, and either d or f is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined as

$$\begin{aligned}\phi(x) &= du_2 + v_1 + \frac{a}{2}v_2,\\ \phi(y) &= u_1 + bu_2,\\ \phi(z) &= fu_2 + \frac{e}{2}v_2 + w,\end{aligned}$$

where by Corollary 1.8 we can choose a primitive element w of $E_8(2)$ with $w^2 = 2c$, $c < 0$. This is an embedding and by Lemma 1.2, it is primitive. Assume that $s = x_1u_1 + x_2u_2 + x_3v_1 + x_4v_2 + w'$ is an element of orthogonal complement of the Imϕ. Then $s \cdot \phi(x) = dx_1 + ax_3 + 2x_4 = 0$, $s \cdot \phi(z) = fx_1 + ex_3 + ww' = 0$. Since d or f is odd, x_1 is even. Thus $s \cdot s \equiv 0 \pmod 4$ and hence cannot be -2.

3-ii) Only a and c are even.
We use the base change by $\left(\begin{smallmatrix}1&1&1\\0&1&0\\0&0&1\end{smallmatrix}\right)$, where $a' = a + b + c + d + e + f$, $b' = b$, $c' = c$, $d' = 2b + d + f$, $e' = 2c + e + f$, and $f' = f$. Then this case is reduced to Case 3-i).

4-i) a, b, and d are even, c is odd, and either e or f is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined as

$$\begin{aligned}\phi(x) &= eu_2 + v_1 + \frac{a}{2}v_2,\\ \phi(y) &= fu_2 + \frac{d}{2}v_2 + w,\\ \phi(z) &= u_1 + cu_2,\end{aligned}$$

where by Corollary 1.8 we can choose a primitive element w of $E_8(2)$ with $w^2 = 2b$, $b < 0$. This is an embedding and by Lemma 1.2, it is primitive. Assume that $s = x_1u_1 + x_2u_2 + x_3v_1 + x_4v_2 + w'$ is an element of orthogonal complement of the Imϕ. Then $s \cdot \phi(x) = ex_1 + ax_3 + 2x_4 = 0$, $s \cdot \phi(y) = fx_1 + dx_3 + ww' = 0$. Since e or f is odd, x_1 is even. Thus $s \cdot s \equiv 0 \pmod 4$ and hence cannot be -2.

4-ii) Only a and b are even.
We use the base change by $\left(\begin{smallmatrix}1&0&1\\0&1&0\\0&0&1\end{smallmatrix}\right)$, where $a' = a + c + e$, $b' = b$, $c' = c$, $d' = d + f$, $e' = 2c + e$, and $f' = f$. Then this case is reduced to Case 4-i).

5) Only a and f are even.
Since we assume that $b, c < 0$, we split into two cases.

i) $f \geq 0$.
We use the base change by $\left(\begin{smallmatrix}1&0&0\\0&1&-1\\0&0&1\end{smallmatrix}\right)$, where $a' = a$, $b' = b + c - f$, $c' = c$, $d' = d - e$, $e' = e$, and $f' = -2c + f$. Then this case is reduced to Case 4-i).

ii) $f < 0$.
We use the base change by $\left(\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right)$, where $a' = a$, $b' = b + c + f$, $c' = c$, $d' = d + e$, $e' = e$, and $f' = 2c + f$. Then this case is reduced to Case 4-i).

6) Only b and e are even.
We use the base change by $\begin{pmatrix} 1&1&0\\ 0&1&0\\ 0&0&1 \end{pmatrix}$, where $a' = a+b+d$, $b' = b$, $c' = c$, $d' = 2b+d$, $e' = e+f$, and $f' = f$. Then this case is reduced to Case 4-ii).

7) Only c and d are even.
Since we assume that $b, c < 0$, we split into two cases.

i) $f \geq 0$.
We use the base change by $\begin{pmatrix} 1&0&1\\ 0&1&-1\\ 0&0&1 \end{pmatrix}$, where $a' = a+c+e$, $b' = b+c-f$, $c' = c$, $d' = -2c+d-e+f$, $e' = 2c+e$, and $f' = -2c+f$. Then a', b', c', and d' are even. Thus this case is reduced to Case 1.

ii) $f < 0$.
We use the base change by $\begin{pmatrix} 1&0&1\\ 0&1&1\\ 0&0&1 \end{pmatrix}$, where $a' = a+c+e$, $b' = b+c+f$, $c' = c$, $d' = 2c+d+e+f$, $e' = 2c+e$, and $f' = 2c+f$. Then a', b', c', and d' are even. Thus this case is reduced to Case 1. □

3. Proof of Theorem 1.4

Let $\{x, y, z\}$ be a basis of the transcendental lattice T_X and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard bases of U and $U(2)$, respectively. We derive a contradiction if an embedding of T_X into Λ^- exists.

1) a, b, and c are even; def is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as

$$(2)\qquad \begin{cases} \phi(x) = a_1u_1 + a_2u_2 + a_3v_1 + a_4v_2 + w_1, \\ \phi(y) = b_1u_1 + b_2u_2 + b_3v_1 + b_4v_2 + w_2, \\ \phi(z) = c_1u_1 + c_2u_2 + c_3v_1 + c_4v_2 + w_3, \end{cases}$$

where the a_i's, b_i's, and c_i's are integers, $w_i \in E_8(2)$. Assume that ϕ is an embedding, i.e.,

$$(3)\qquad \begin{cases} \phi(x)\cdot\phi(x) = 2a_1a_2 + 4a_3a_4 + w_1^2 = 2a, \\ \phi(y)\cdot\phi(y) = 2b_1b_2 + 4b_3b_4 + w_2^2 = 2b, \\ \phi(z)\cdot\phi(z) = 2c_1c_2 + 4c_3c_4 + w_3^2 = 2c, \\ \phi(x)\cdot\phi(y) = a_1b_2 + a_2b_1 + 2a_3b_4 + 2a_4b_3 + w_1w_2 = d, \\ \phi(x)\cdot\phi(z) = a_1c_2 + a_2c_1 + 2a_3c_4 + 2a_4c_3 + w_1w_3 = e, \\ \phi(y)\cdot\phi(z) = b_1c_2 + b_2c_1 + 2b_3c_4 + 2b_4c_3 + w_2w_3 = f. \end{cases}$$

Since a is even and d is odd, either a_1 or a_2 is even; similarly for b_1, b_2 and c_1, c_2. Without loss of generality, we may assume that a_1 is even. Then since d and e are odd a_2, b_1, and c_1 are odd. Hence b_2 and c_2 are even. Then f is even which is a contradiction, so T_X has no embedding into Λ^-.

2) af is odd; b, c, and de are even.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since a is odd and b, c are even, from (3) a_1, a_2 are odd, b_1 or b_2 is even, and c_1 or c_2 is even. Also, since f is odd, $\left(\begin{smallmatrix} b_1 \text{ and } c_2 \text{ are even} \\ b_2 \text{ and } c_1 \text{ are odd} \end{smallmatrix}\right)$ or

$\left(\begin{smallmatrix} b_2 \text{ and } c_1 \text{ are even} \\ b_1 \text{ and } c_2 \text{ are odd} \end{smallmatrix}\right)$. Since d or e is even, b_1, b_2 are even or c_1, c_2 are even. So f is even. Hence, T_X has no embedding into Λ^-.

3) be is odd; a, c, and df are even.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since b and e are odd, from (3) $b_1, b_2, a_1c_2 + a_2c_1$ are odd. Also, since a, c are even, $\left(\begin{smallmatrix} a_1 \text{ and } c_2 \text{ are even} \\ a_2 \text{ and } c_1 \text{ are odd} \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a_2 \text{ and } c_1 \text{ are even} \\ a_1 \text{ and } c_2 \text{ are odd} \end{smallmatrix}\right)$. Then, both d and f are odd. Hence, T_X has no embedding into Λ^-.

4) cd is odd; a, b, and ef are even.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since c and d are odd, from (3) $c_1, c_2, a_1b_2 + a_2b_1$ are odd. Also, since a, b are even, $\left(\begin{smallmatrix} a_1 \text{ and } b_2 \text{ are even} \\ a_2 \text{ and } b_1 \text{ are odd} \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} a_2 \text{ and } b_1 \text{ are even} \\ a_1 \text{ and } b_2 \text{ are odd} \end{smallmatrix}\right)$. Then, both e and f are odd. Hence, T_X has no embedding into Λ^-.

5-i) a is even and bcf is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since b and c are odd, from (3) b_1, b_2, c_1, c_2 are odd. Then, f is even. Hence, T_X has no embedding into Λ^-.

5-ii) a and f are even; b, c, and $d + e$ are odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Without loss of generality, we may assume that d is odd. Then, either a_1 or a_2 is even. Since b_1, b_2, c_1, and c_2 are odd, e is also odd. That is, d and e have the same sign. Hence, T_X has no embedding into Λ^-.

6-i) b is even and ace is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since a and c are odd, from (3) a_1, a_2, c_1, and c_2 are odd. Then, e is even. Hence, T_X has no embedding into Λ^-.

6-ii) b and e are even; a, c, and $d + f$ are odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Without loss of generality, we may assume that d is odd. Then, either b_1 or b_2 is even. Since a_1, a_2, c_1, and c_2 are odd, f is also odd. That is, d and f have the same parity. Hence, T_X has no embedding into Λ^-.

7-i) c is even and abd is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since a and b are odd, from (3) a_1, a_2, b_1, and b_2 are odd. Then, d is even. Hence, T_X has no embedding into Λ^-.

7-ii) c and d are even; a, b, and $e + f$ are odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Without loss of generality, we may assume that e is odd. Then, either c_1 or c_2 is even. Since a_1, a_2, b_1, and b_2 are odd, f is also odd. That is, e and f have the same parity. Hence, T_X has no embedding into Λ^-.

8) a, b, and c are odd; d, e, or f is odd.
Consider the mapping $\phi : T_X \longrightarrow \Lambda^-$ defined generically as in (2). Assume that ϕ is an embedding. Since a, b, and c are odd, from (3) a_1, a_2, b_1, b_2, c_1, and c_2 are odd. Then, d, e, and f are even. Hence, T_X has no embedding into Λ^-. □

4. Remaining cases

The remaining cases are as follows:
1. Only a is odd.
2. Only b is odd.
3. Only c is odd.
4. Only a and b are odd.
5. Only b and c are odd.
6. Only a and c are odd.
7. Only a, b, and c are odd.

However, these are all equivalent. First, we show that Cases 4 and 7 are equivalent to Case 2 and that Cases 5 and 6 are equivalent to Case 3.

Lemma 4.1. *The case in which only a and b are odd is equivalent to the case in which only b is odd.*

Proof. We use the base change by $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $a' = a + b + c + d + e + f$, $b' = b$, $c' = c$, $d' = 2b + d + f$, $e' = 2c + e + f$, and $f' = f$. Then this case is reduced to the case in which only b is odd. □

Lemma 4.2. *The case in which only a, b, and c are odd is equivalent to the case in which only b is odd.*

Proof. Since we assume that $b, c < 0$, we split into two cases.
i) $f \geq 0$.
We use the base change by $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, where $a' = a + 4b + c + 2d + e + 2f$, $b' = b$, $c' = b + c - f$, $d' = 4b + d + f$, $e' = -4b + 2c - d + e + f$, and $f' = -2b + f$. Then this case is reduced to the case in which only b is odd.
ii) $f < 0$.
We use the base change by $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, where $a' = a + 4b + c + 2d + e + 2f$, $b' = b$, $c' = b + c + f$, $d' = 4b + d + f$, $e' = 4b + 2c + d + e + 3f$, and $f' = 2b + f$. Then this case is reduced to the case in which only b is odd. □

Lemma 4.3. *The case in which only b and c are odd is equivalent to the case in which only c is odd.*

Proof. Since we assume that $b, c < 0$, we split into two cases.
i) $f \geq 0$.
We use the base change by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, where $a' = a$, $b' = b + c - f$, $c' = c$,

$d' = d - e$, $e' = e$, and $f' = -2c + f$. Then this case is reduced to the case in which only c is odd.

ii) $f < 0$.

We use the base change by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, where $a' = a$, $b' = b + c + f$, $c' = c$, $d' = d + e$, $e' = e$, and $f' = 2c + f$. Then this case is reduced to the case in which only c is odd. □

Lemma 4.4. *The case in which only a and c are odd is equivalent to the case in which only c is odd.*

Proof. We use the base change by $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $a' = a + c + e$, $b' = b$, $c' = c$, $d' = d + f$, $e' = 2c + e$, and $f' = f$. Then this case is reduced to the case in which only c is odd. □

Now in order to show that Cases 1, 2, and 3 are equivalent we prove the following lemma.

Lemma 4.5. *Let T_X be a lattice given as* (1) *with $b, c < 0$. Then T_X is $\mathbb{Z}$-equivalent to T'_X with $a', b', c' < 0$.*

Proof. Assume that $a \geq 0$ (If $a < 0$, there is nothing to prove). Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Since $A \in SL_3(\mathbb{Z})$, the matrix AT_XA^{tr} is $\mathbb{Z}$-equivalent to T_X. Now $T''_X = AT_XA^{tr} = \begin{pmatrix} 2a'' & d'' & e'' \\ d'' & 2b'' & f'' \\ e'' & f'' & 2c'' \end{pmatrix}$, where $a'' = c$, $b'' = a$, $c'' = b$, $d'' = e$, $e'' = f$, $f'' = d$. Then $c'' < 0$. By Lemma 1.6, T''_X is $\mathbb{Z}$-equivalent to T'_X with $b', c' < 0$ and $a'' = a'$. That is, T_X is $\mathbb{Z}$-equivalent to T'_X with $a', b', c' < 0$. □

Now we can assume that a, b, c of Cases 1, 2, and 3 are negative.

Consider the base change by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, where $a' = b$, $b' = c$, $c' = a$, $d' = f$, $e' = d$, and $f' = e$. Then Case 2 is reduced to Case 1. Also the base change by $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, where $a' = c$, $b' = a$, $c' = b$, $d' = e$, $e' = f$, $f' = d$, forces Case 3 to be Case 1.

Remark 4.6. Whether a K3 surface covers an Enriques surface or not certainly does not depend on the choice of a basis for the transcendental lattice. For some choice of such a basis, if the parities of the integers in T_X satisfy one of the conditions in Theorem 1.3, then the parities of T_X with respect to any other basis will again satisfy one of the, possibly different, conditions of Theorem 1.3. And the same holding for T_X satisfying the conditions of Theorem 1.4.

Now we only consider the case in which only a is odd. In this case we do not exactly know whether the K3 surface cover an Enriques surface. However, there is a partial solution using the spinor genus of an indefinite ternary quadratic form.

Theorem 4.7 (Eichler, [2]). *For indefinite forms of dimension of at least* 3*, a spinor genus contains exactly one integral equivalence class of forms.*

Thus if the spinor genus of the remaining case is the same as one case of Theorem 1.3 and Theorem 1.4, then we know whether the K3 surface is a K3 cover of some Enriques surface.

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References

[1] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, 2004.

[2] M. Eichler, *Quadratische formen und orthogonal gruppen*, Springer-Verlag, 1952.

[3] J. Keum, *Every algebraic Kummer surface is the K3-cover of an Enriques surface*, Nagoya Math. J. **118** (1990), 99–110.

[4] V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Math. USSR Izvestija **14** (1980), no. 1, 103–167.

[5] A. Sertöz, *Which singular K3 surfaces cover an Enriques surface*, Proc. Amer. Math. Soc. **133** (2005), no. 1, 43–50.

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