ON CONJUGACY OF \( p \)-GONAL AUTOMORPHISMS

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Abstract. In 1995 it was proved by González-Diez that the cyclic group generated by a \( p \)-gonal automorphism of a closed Riemann surface of genus at least two is unique up to conjugation in the full group of conformal automorphisms. Later, in 2008, Gromadzki provided a different and shorter proof of the same fact using the Castelnuovo-Severi theorem. In this paper we provide another proof which is shorter and is just a simple use of Sylow’s theorem together with the Castelnuovo-Severi theorem. This method permits to obtain that the cyclic group generated by a conformal automorphism of order \( p \) of a handlebody with a Kleinian structure and quotient the three-ball is unique up to conjugation in the full group of conformal automorphisms.

1. \( p \)-gonal automorphisms of Riemann surfaces

In [3] it was proved by González-Diez that the cyclic group generated by a \( p \)-gonal automorphism of a closed Riemann surface of genus at least two is unique up to conjugation in the full group of conformal automorphisms. Later, Gromadzki [4] provided a different and shorter proof of the same fact using the Castelnuovo-Severi theorem. In this paper we provide another proof which is shorter and which is just a simple use of Sylow’s theorem together with the Castelnuovo-Severi theorem. If \( S \) is a Riemann surface, then we denote by \( \text{Aut}(S) \) the group of conformal automorphisms of \( S \).

Theorem 1 (González-Diez [3], Gromadzki [4]). Let \( S \) be a given closed Riemann surface of genus \( g \geq 2 \) and let \( \phi \in \text{Aut}(S) \) be a conformal automorphism of order a prime \( p \) so that \( S/\langle \phi \rangle \) is the Riemann sphere \( \hat{C} \). Then the cyclic group \( \langle \phi \rangle \) is unique up to conjugation in \( \text{Aut}(S) \).

Proof. If \( p = 2 \), then \( \phi \) is the hyperelliptic involution, which is known to be unique (see, for instance, [2]). So, we may assume \( p \geq 3 \). Let us consider a branched regular cover \( \pi : S \to \hat{C} \) with deck group being \( \langle \phi \rangle \). If we denote by \( k \) the number of fixed points of \( \phi \), then the Riemann-Hurwitz formula asserts

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that $g = (p - 1)(k - 2)/2$. The Castelnuovo-Severi theorem [1] asserts that if $g > (p - 1)^2$, then $\langle \phi \rangle$ is unique in $\text{Aut}(S)$. So, we may also assume that $g \leq (p - 1)^2$; in particular $k \leq 2p$. If $\langle \phi \rangle$ is a $p$-Sylow subgroup of $\text{Aut}(S)$, then Sylow's theorem asserts that $\langle \phi \rangle$ is unique up to conjugation in $\text{Aut}(S)$.

We assume from now on that this is not the case. In particular, by Sylow’s theorem, there is a $p$-subgroup $K$ of order $p^2$ (so an abelian group) containing $\langle \phi \rangle$ as a normal subgroup. The group $K$ induces a conformal automorphism $\psi$ of $\hat{\mathbb{C}}$ (so a Möbius transformation) of order $p$ that permutes the $k$ branch values of $\pi$. As $\psi$ has exactly two fixed points, then $k = a + lp$, where $a \in \{0, 1, 2\}$ and $l \in \{0, 1, \ldots \}$. As $0 < k \leq 2p$, there are exactly two cases to consider: (i) $a = 0$ and $l \in \{1, 2\}$ or (ii) $a \neq 0$ and $l \in \{0, 1\}$.

If $a \neq 0$, then it follows that $\psi$ fixes some of the branch values of $\pi$. This asserts that there is a fixed point of $\phi$ which is also a fixed point of every element in $K$. In this way, $K = \langle \eta \rangle \cong \mathbb{Z}_p^2$ and $\phi = \eta^p$. If $K$ is a $p$-Sylow subgroup, then $\langle \phi \rangle$ is unique up to conjugation. If $K$ is not a $p$-Sylow subgroup, then there is a $p$-subgroup $L$ of order $p^3$, so that $K$ is a normal subgroup of $L$, in particular, $\langle \phi \rangle$ is also a normal subgroup of $L$. This asserts that $L$ induces a group $H$ of order $p^2$ (so abelian) of conformal automorphisms of $\hat{\mathbb{C}}$ (so a finite abelian group of Möbius transformations) keeping invariant the branch values of $\pi$. As the only abelian groups of Möbius transformations are either isomorphic to $\mathbb{Z}_2$ or to $\mathbb{Z}_n$ and $p \geq 3$, it follows that $H = \langle \rho \rangle \cong \mathbb{Z}_p$. As $H$ contains $\psi$, we may also assume that $\rho^p = \psi$. It follows that the two fixed points of $\rho$ are the same as for $\psi$. As the branch values of $\pi$ are permuted by the rotation $\rho$ of order $p^2$, at least one of the fixed points of $\eta$ is also a branch value of $\pi$, and $k \leq p^2$, we must have $k = a$. As $a \in \{1, 2\}$, this contradicts the assumption that $g \geq 2$.

Let us now assume $a = 0$. In this case, we have that $k \in \{p, 2p\}$. If $k = p$, then $g = (p - 1)(p - 2)/2$ and we may assume that $\psi(z) = e^{2\pi i/p}z$ and that the $k$ branch points are given by the $p$-roots of unity. In this case, $S$ is given by the Fermat curve $y^p + x^p + z^p = 0$ and $\phi(x, y, z) = (x, e^{2\pi i/p}y, z)$. It is well known that for this surface $\text{Aut}(S) \cong \mathbb{Z}_p^2 \rtimes \mathfrak{S}_3$ ($\mathfrak{S}_3$ is the symmetric group of order 6), where $\mathbb{Z}_p^2$ is generated by $\phi$ and $\tau(x, y, z) = (e^{2\pi i/p}x, y, z)$ and $\mathfrak{S}_3$ is generated by $\rho(x, y, z) = (y, x, z)$ and $\eta(x, y, z) = (y, z, x)$. The cyclic groups or order $p$ generated by an automorphism with fixed points are either $\langle \phi \rangle$, $\langle \tau \rangle$ or $\langle \phi \tau \rangle$. These three subgroups are conjugated by $\mathfrak{S}_3$.

If $k = 2p$, then $g = (p - 1)^2$ and again we may assume that $\psi(z) = e^{2\pi i/p}z$ and that the $k$ branch points are given by the $p$-roots of unity and the $p$-roots of some $a^p \neq 0, 1$. In this case, $S$ is given by the algebraic curve $y^p = (x^p - 1)(x^p - a^p)^r$, where $r \in \{1, \ldots, p - 1\}$, and $\phi(x, y) = (x, e^{2\pi i/p}y)$. The group $H \cong \mathbb{Z}_p^2 \rtimes L$, where $\mathbb{Z}_p^2$ is generated by $\phi$ and $\tau(x, y) = (e^{2\pi i/p}x, y)$, and that $L$ is either isomorphic to either the trivial group or $\mathbb{Z}_2 \rtimes D_1$ depending on the value of $r$, is a subgroup of $\text{Aut}(S)$ (see also [8]). The quotient orbifold $S/H$ has either signature $(2, 2, 2, p)$ or $(2, 4, 2p)$. By Singerman’s list of not finitely maximal Fuchsian groups [7], we may see that the above two signatures
are maximal ones. In particular, it asserts that $\text{Aut}(S) = H$. In this case, the cyclic groups of order $p$ generated by automorphisms with fixed points are either $\langle \phi \rangle$ or $\langle \tau \rangle$. If $r < p-1$, then it follows (from the Riemann-Hurwitz formula) that $S/\langle \tau \rangle$ has genus greater than zero. If $r = p-1$, then $\langle \phi \rangle$ and $\langle \tau \rangle$ are conjugated by the automorphism $\alpha(x, y) = \left( (x^p - a^p)/y, x^p-1(1-a^p)/(x^p - 1) \right)$ [3]. □

2. $p$-gonal automorphisms of handlebodies

A Kleinian group is a discrete group of Möbius transformations and its region of discontinuity is the set of points on $\mathbb{C}$ on which the group acts discontinuously. By the Poincaré extension property, every Kleinian group acts on the hyperbolic 3-space $\mathbb{H}^3$. If $\Gamma$ is a Kleinian group, with region of discontinuity $\Omega$, then $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ is called a Kleinian 3-orbifold uniformized by $\Gamma$. The conformal boundary $S = \Omega/\Gamma$ is a Riemann orbifold and the interior $M^0 = \mathbb{H}^3/\Gamma$ is an hyperbolic 3-orbifold. If $\Gamma$ is torsion free, then $M$ is a Kleinian 3-manifold, $S$ is a Riemann surface and $M^0$ is an hyperbolic 3-manifold.

If $\Gamma$ is torsion free, then a conformal automorphism of $M$ is a self-homeomorphism $\phi : M \to M$ so that its restriction $\phi : M^0 \to M^0$ is a hyperbolic isometry (the restriction $\phi : S \to S$ is a conformal automorphism). We denote by $\text{Aut}(M)$ the group of conformal automorphisms of $M$. If $H < \text{Aut}(M)$, then, by lifting to the universal cover space $\mathbb{H}^3 \cup \Omega$, one obtains a Kleinian group $G$ containing $\Gamma$ as a normal subgroup so that $G = \langle \phi \rangle$.

If $\Gamma$ has torsion, then one can also define conformal automorphisms of $M$, but in this case one has to ensure the self-homeomorphism $\phi$ to preserve the cone set of $M$ (the projection of points with non-trivial $\Gamma$-stabilizers) in order to ensure that it can be lifted to obtain a Kleinian group $G$ containing $\Gamma$ as a normal subgroup so that $G/\Gamma = \langle \phi \rangle$ (this is similar to the case of Riemann orbifolds).

Let us assume $M$ is a handlebody of genus $g \geq 2$. It is well known that in this case $\Gamma$ is a Schottky group of rank $g$. As a handlebody is a retraction body, one can notice that if $\phi_1, \phi_2 \in \text{Aut}(M)$ are so that $\phi_1|_{S} = \phi_2|_{S}$, then $\phi_1 = \phi_2$. In this way, there is a natural one-to-one homomorphism $\theta : \text{Aut}(M) \to \text{Aut}(S)$ given by restriction. In general, this homomorphism is not surjective. Necessary and sufficient conditions for an element of $\text{Aut}(S)$ to be in the image of $\theta$ can be found in [5].

Let $\phi_1, \phi_2 \in \text{Aut}(M)$ be a conformal automorphisms of order a prime $p$ so that $M/\langle \phi_j \rangle$ is the 3-ball. If we consider the restrictions to $S$, we known from González-Diez’s result the existence of some $\tau \in \text{Aut}(S)$ so that $\tau(\phi_1) \tau^{-1} = \langle \phi_2 \rangle$. As $\theta$ is not surjective in general, it may be that $\tau$ is not the restriction of an element of $\text{Aut}(M)$. Nevertheless, using the same ideas of the proof above, we notice that González-Diez’s result still valid at the level of handlebodies.

**Theorem 2.** Let $M$ be a given handlebody with a Kleinian structure of genus $g \geq 2$ and let $\phi \in \text{Aut}(M)$ be a conformal automorphism of order a prime $p$ so
that $M/\langle \phi \rangle$ is the 3-ball. Then the cyclic group $\langle \phi \rangle$ is unique up to conjugation in $\text{Aut}(M)$.

Proof. Let $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ be a handlebody of genus $g \geq 2$, uniformized by the Schottky group $\Gamma$, and let $\phi \in \text{Aut}(M)$ of order $p$ prime so that $B = M/\langle \phi \rangle$ is the 3-ball. By lifting $\phi$ to the universal cover space, we obtain a Kleinian group $G$ containing $\Gamma$ as a normal subgroup of index $p$ and $G/\Gamma = \langle \phi \rangle$. Clearly, $B = (\mathbb{H}^3 \cup \Omega)/G$.

We may now follow the same proof as in the previous case by working at the level of the restriction of $\phi$ to $S = \Omega/\Gamma$. We may assume $p > 2$.

First, we notice that the injectivity of $\theta$ asserts that if $\langle \phi \rangle$ is unique in $\text{Aut}(S)$, then $\langle \phi \rangle$ is unique in $\text{Aut}(M)$. So, by Castelnuovo-Severi's theorem, we may assume $g \geq (p - 1)^2$. Again, we may also assume that $\langle \phi \rangle$ is a not a $p$-Sylow subgroup of $\text{Aut}(M)$. So, we have $K < \text{Aut}(M)$ of order $p^2$ so that $\langle \phi \rangle$ is a normal subgroup of $K$. Again working at the level of $S$, one obtains that $k = a + lp \leq 2p$, where either (i) $a = 0$ and $l \in \{1, 2\}$ or (ii) $a \neq 0$ and $l \in \{0, 1\}$. In the case $a \neq 0$ we again obtain that $K \cong \mathbb{Z}_{p^2}$ must be a $p$-Sylow subgroup of $\text{Aut}(M)$ and, in particular, that $\langle \phi \rangle$ is unique up to conjugation.

If $a = 0$, then we have that $k \in \{p, 2p\}$. But, the number $k$ of fixed points of $\phi$ on $S$ is necessarily even [5]. It follows that $k = 2p$. Moreover, the cone set of $B$ is given by $p$ pairwise disjoint simple arcs, which are cyclically permuted by the automorphism $\psi$ (again induced by $K$). In this case, we have that $S$ is given by the algebraic curve $y^p = (x^p - 1)(x^p - a^p)r$, where $r \in \{1, \ldots, p - 1\}$, and $\phi(x, y) = (x, e^{2\pi i/p}y)$. We have, as already seen, that the only case to consider is $k = p - 1$. But in this case, the quotient orbifold $M/A$, where $A = \langle \phi, \tau \rangle \cong \mathbb{Z}_p^2$, is the 3-ball with exactly two disjoint simple arcs. We only need to ensure that the automorphism $a(x, y) = ((x^p - a^p)/y, x^{p-1}(1 - a^p)/(x^p - 1)) \in \text{Aut}(S)$ belongs to the image of $\theta$. For it, let us assume the end points of the two cone arcs of $M/A$ are $\infty, 0$ (for the first one) and $1$ and $\lambda \in \mathbb{C} - \{0\}$ (for the second one), where we have identified the conformal boundary of $M/A$ with $\hat{\mathbb{C}}$. We notice that the Möbius transformation $\beta(z) = \lambda(z - 1)/(z - \lambda)$ induces a conformal automorphism of $M/A$. This automorphism necessarily lifts to $M$ to a conformal automorphism of order two that conjugates $\langle \phi \rangle$ to $\langle \tau \rangle$. More explicitly, the Kleinian group $K_1$ uniformizing the orbifold $(M/A)/\langle \beta \rangle$ is a free product (in the sense of the Klein-Maskit combination theorems [6]) of a cyclic group generated by an elliptic transformation $\gamma$ of order $p$ and a cyclic group generated by an elliptic transformation $\eta$ of order two. The index two subgroup $K_2 = \langle \gamma, \eta, \gamma \eta \rangle \cong \mathbb{Z}_p * \mathbb{Z}_p$ uniformizes $M/A$. The group $\Gamma$ is the smallest normal subgroup of $K_2$ containing the commutator $[\gamma, \eta \gamma \eta]$. The transformation $\gamma$ induces $\phi$, and $\eta \gamma \eta$ induces $\tau$. It can be seen by direct computation that $\Gamma$ is also a normal subgroup of $K_1$. In this way, the transformation $\eta$ induces a conformal involution on $M$ conjugating $\langle \phi \rangle$ to $\langle \tau \rangle$. \hfill \Box

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References


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