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ON VON NEUMANN REGULAR RINGS, VIII

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INTRODUCTION. Von Neumann regular and associated rings (for example, V-rings) are extensively studied since several years (cf. [2], [3]). It is well-known that A is von Neumann regular iff every right (left) A-module is flat. Up to now, we have considered von Neumann regularity essentially through P-injective and certain flat modules (cf. [7] to [9], [11] to [16]). (For completeness, recall that a left A-module M is p-injective iff for any principal left ideal P of A, any left A-homomorphism of P into M extends to one of A into M.) In this note, generalizations of injectivity, noted CF and MP-injectivity, are introduced and connections between injectivity, CF-injectivity, MR-injectivity and von Neumann regularity are found. If C is a cyclic left A-module whose submodules are cyclic, it is proved that ${}_AC$ is injective iff it is CF-injective. A is a left self-injective ring iff every essential extension of ${}_AA$ is CF-injective. It is well known that over left self-injective rings, the Jacobson radical coincides with the left singular ideal (*cf.* [2, p. 78]). This is here generalized to left MP-injective rings. Left self-injective regular rings may be characterized as left non-singular rings whose finitely generated non-singular left modules are CF-injective. A is semi-simple Artinian iff every finitely generated semi-simple left A-module is CF-injective iff every essentially cyclic left A-module is CF-injective. The following interesting characterization of division rings will yield a positive answer to a question raised in [13] : A is a division ring iff A is left uniform such that each simple right A-module is flat.

Throughout, A represents an associative ring with identity and A-modules are unitary. J, Z will denote respectively the Jacobson radical and the left singular ideal of A. A is called left non-singular iff $Z=0$. We introduce the following generalizations of injectivity:

DEFINITIONS. (1) A left A-module M is called CF-injective if, for any finitely generated left A-module F, any cyclic left submodule C of F, every left A-homomorphism of C into M extends to one of F into M.

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(2) ${}_AM$ is called MP-injective if, for any principal left ideal P of A, any left A-monomorphism of P into M extends to a left A-homomorphism of A into M.

A is called left CF-injective (resp. MP-injective) iff ${}_AA$ is CF-injective (resp. MP-injective). Any direct summand of a CF-injective (resp. MP-injective) left A-module is CF-injective (resp. MP-injective). Also, a finite direct sum of CF-injective left A-modules is CF-injective.

Obviously, CF-injectivity implies MP-injectivity but the converse is not true. Indeed, if A is semi-prime, then it may be easily shown that any simple left (or right) A-module is MP-injective. Since rings whose simple left modules are p-injective must be fully left idempotent (cf. [8, Proposition 6]), then MP-injectivity effectively generalises p-injectivity (even in the commutative case) and hence CF-injectivity.

Note that quasi-injective modules need not be CF-injective (otherwise, any left Artinian ring would be completely reducible!).

The proof of [2, Theorem 19.27] yields the following CF-injective analogue of a well-known result of C. Faith - Y. Utumi concerning quasi-injective modules.

THEOREM 1. *Let M be a cyclic CF-injective left A-module such that every complement left submodule is cyclic. If $E=\text{End}\,({}_AM)$. then $E/J(E)$ is von Neumann reqular, where $J(E)=\{f\in E/\ker f$ is essential in ${}_AM\}$ is the Jacobson radical of E.*

We now give a sufficient condition for a cyclic CF-injective left A-module to be injective.

PROPOSITION 2. *Let C be a cyclic left A-module such that every left submodule is cyclic. Then ${}_AC$ is injective iff it is CF-injective.*

Proof. One implication is obvious. Therefore suppose ${}_AC$ is CF-injective. Let L be an essential left ideal of A, $i: L\to A$ the inclusion map, $f: L\to C$ any non-zero left A-homomorphism. It is sufficient to prove that f extends to a left A-homomorphism of A into C and then ${}_AC$ will be injective. If $K=L/\ker f$, $T=A/\ker f$, $P: L\to K$ and $r: A\to T$ the natural projections, $j: K\to T$ the inclusion map, $g: K\to C$ the left A-homomorphism induced by f, then g yields an isomorphism of K onto $\text{Im} f$ and by hypothesis, ${}_AK$ is cyclic. The set E of left submodules M of T containing K such that g extends to a left A-homomorphism of M into C is non-empty (because $K\in E$) and by Zorn's Lemma, E has a maximal member U. Let $h: U\to C$ be the left A-homomorphism which extends g. If $U\neq T$, let $y\in T$, $y\notin U$. Since

${}_AU/\mathrm{ker}h$ is cyclic by hypothesis, then with $\bar{y}=y+\mathrm{ker}h$, $F=U/\mathrm{ker}h+A\bar{y}$ is a finitely generated left A-submodule of $T/\mathrm{ker}h$, whence by the CF-injectivity of C. the left A-homomorphism h' of $U/\mathrm{ker}h$ into C induced by h extends to a left A-homomorphism k of F into C. If $s' : U\to U+Ay$, $t' : U/\mathrm{ker}h\to F$ are the inclusion map, $p' : U\to U/kerh$ and $r' : U+A_y\to F$ the natural projections, then $h=h'p'=kt'p'=(kr')s'$ which proves that kr'extends h(and hence g) to $U+Ay$, whence $U+Ay\in E$, contradicting the maximality of U. Therefore $U=T$ and since $f=gp$, $jp=ri$, then $f=hri$ which proves that f extends to $hr : A\to C$.

COROLLARY 2. 1. *A cyclic completely reducible left A-module is injective iff it is CF-injective. In particular, if the left socle S of A is CF-injective, then any principal left ideal contained in S is injective. Consequently, if A is of left finite Goldie dimension, then the left socle is injective iff it is CF-injective.*

COROLLARY 2. 2. *A is a left V-ring iff every simple left A-module is CF-injective.*

A result of Y. Utumi (cf. [2, Corollary 19. 28(a)]) is improved in the next proposition. Following [2], an element c of A is called left regular iff $1(c)=0$. Then c is a non-zero-divisor iff c is left and right regular.

PROPOSITION 3. *Let A be a left MP-injective ring. Then* (1) *Any left regular element of A is right invertible. Consequently, every left (or right) A-module is divisible*; (2) $Z=J$.

Proof. (1) If $c\in A$ is left regular, $g : Ac\to A$ the left A-homomorphism defined by $g(ac)=a$ for all $a\in A$ then g is a monomorphism which implies the existence of $y\in A$ such that $1=g(c)=cy$. Consequently, if c is a non-zero-divisor, then $c=cyc$ implies $1-yc=0$, whence c is invertible in A. Therefore $M=cM$ for every left A-module M and similarly, every right A-module is divisible.

(2) Let $z\in Z$. For any $a\in A$, if $u\in \mathrm{l}\ (1-za)$, then $u=uza$ implies $Au\cap \mathrm{l}(za)=0$, whence $u=0$ (since $za\in Z$). By (1), $(1-za)y=1$ for some $y\in A$ which proves that $z\in J$. Now suppose there exists $w\in J$ such that $w\notin Z$. Then there exists a non-zero complement left ideal K such that $L=\mathrm{l}(w)\oplus K$ is an essential left ideal. For any $0\neq k\in K$, $kw\neq 0$ and if $f : Akw\to A$ is the left A-homomorphism defined by $f(akw)=ak$ for all $a\in A$, then f is a monomorphism and there exists $d\in A$ such that $k=f(kw)=kwd$. Now $(1-wd)\ v=1$ for some $v\in A$ (since $wd\in J$) which yields $k=k\ (1-wd)v=(k-kwd)v=0$, a contradiction. This proves that $Z=J$.

Before characterizing von Neumann regular rings in terms of *MP*-injectivity, let us note the following

REMARK 1. If A is left *CF*-injective such that every complement left ideal is principal, then A/Z is von Neumann regular. (*cf.* Theorem 1)

REMARK 2. If I is a *MP*-injective left ideal of A, then A/I is a flat left A-module. It then follows that a finitely generated *MP*-injective left ideal is a direct summand of ${}_AA$.

REMARK 3. If A is a prime left *MP*-injective ring, then the centre of A is a field.

REMARK 4. A prime left *MP*-injective ring whose essential left ideals are ideals is primitive with non-zero socle.
(An ideal will always mean a two-sided ideal.)

REMARK 5. If U is a minimal left ideal of a left *MP*-injective ring A, for any $u \in U$, uA is a right annihilator.

THEOREM 4. *The following conditions are equivalent*:
(1) *A is von Neumann regular*;
(2) *Every left A-module is MP-injective*;
(3) *A is a left MP-injective ring whose principal left ideals are projective*;
(4) *A is left MP-injective such that for any $0 \neq a \in A$, there exists $0 \neq b \in A$ such that $\mathrm{l}(a) \oplus Ab$ is an essential left ideal.*

Proof. Since A is von Neumann regular iff every left A-module is p-injective, then (1) implies (2).
Since a principal *MP*-injective left ideal is a direct summand of ${}_AA$ (Remark 2), then (2) implies (3).
It is clear that (3) implies (4).
Assume (4). Then $Z=0$. If $0 \neq c \in A$, there exists $0 \neq b \in A$ such that $L = \mathrm{l}(c) \oplus Ab$ is an essential left ideal of A. Then $bc \neq 0$ and if $g: Abc \to A$ is the left A-homomorphism defined by $g(abc)=ab$ for all $a \in A$, then g is a monomorphism which implies that $b=g(bc)=bcd$ for some $d \in A$. Now $Ab \subseteq \mathrm{l}(c-cdc)$ which yields $L \subseteq \mathrm{l}(c-cdc)$, whence $c-cdc \in Z=0$. This proves A regular and hence (4) implies (1).

A is called a left *CM*-ring (cf. [14], [15]) if, for any maximal essential left ideal M (if it exists) of A, every complement left subideal is an ideal of M. Continuous regular rings considered here are those of Utumi [10].

Applying [7, Lemma 2] and [15, Lemma 1.1] to Theorem 4(3), we get

COROLLARY 4.1. *The following conditions are equivalent*:
(1) *A is either semi-simple Artinian or a left and right continuous strongly regular ring*;
(2) *A is a left CM, left MP-injective Baer ring.*

The proof of "(4) implies (1)" in Theorem 4 shows the validity of the next result.

THEOREM 5. *The following conditions are equivalent*:
(1) *A is left continuous regular*;
(2) *A is left non-singular left MP-injective such that any principal or complement left ideal is the left annihilator of an element of A*;
(3) *A is a left non-singular left MP-injective ring whose complement left ideals are principal.*

The next remark is motivated by [6, Proposition 1].

Remark 6. A is quasi-Frobeniusean iff A is a left Artinian, left and right MP-injective ring.

REMARK 7. If every left ideal of A not isomorphic to ${}_AA$ is MP-injective, then A is a semi-prime left semi-hereditary ring whose finitely generated left ideals are principal (cf. [12, Lemma 1.1]).

We now give a nice characterization of division rings which will yield a positive answer to a question raised in [13]. As usual, A is called left uniform iff every non-zero left ideal is essential.

THEOREM 6. *The following conditions are equivalent*:
(1) *A is a division ring*;
(2) *A is a left uniform ring whose simple right modules are flat.*

Proof. Obviously, (1) implies (2).

Assume (2). Let $a\in A$, $a\notin Z$. Then $\mathrm{l}(a)=0$ and if we suppose that $aA \neq A$, let R be a maximal right ideal containing aA. Since A/R_A is flat, then for any left ideal I of A, $I\cap R=RI$ and in particular, $a=ba$ for some $b\in R$. Then $1-b\in\mathrm{l}(a)=0$, contradicting $R\neq A$. This proves that any proper right ideal, in particular every maximal right ideal, is contained in Z. whence Z is the unique maximal right ideal of A. A is therefore a local ring and $Z=J$ is also the only maximal left ideal of A. Now suppose that $Z\neq 0$. Then there exists $0\neq z\in Z$ such that $z^2=0$ and since $\mathrm{l}(z)\subseteq Z$, A/Z_A

is flat, then $z=uz$ for some $u\in Z$. Therefore $1-u\in l(z)\subseteq Z$ implies $1\in Z$, which is impossible. Thus $J=Z=0$ which proves that (2) implies (1).

Looking at the Proof of [13, Lemma 9(1)], we may assert that a prime left CM-ring is either simple Artinian or a left uniform ring. Theorem 6 then yields a corollary which contains a positive answer to [13, Question 2[a)].

COROLLARY 6. 1. *A is simple Artinian iff A is a prime left CM-ring whose simple right modules are flat.*

Following [5], a left A-module M is semi-simple iff the intersection of all maximal left submodules of M is zero. Essentially finitely generated modules are considered in [1]. Call a left A-module M essentially cyclic if ${}_AM$ is an essential extension of a cyclic left A-module. Condition (4) below weakens the following well-known characteristic property of semi-simple Artinian rings due to B. Osofsky: Every cyclic left module is injective.

THEOREM 7. *The following conditions are equivalent:*
(1) *A is semi-simple Artinian;*
(2) *Every finitely generated semi-simple left A-module is CF-injective;*
(3) *Every finitely generated left A-module is CF-injective;*
(4) *Every esentially cyclic left A-module is CF-injective;*
(5) *A is a semi-prime left MP-injective ring satisfying the maximum condition on left and right annihilators.*

Proof. (1) implies (2) evidently.
(2) implies (3) by [5, Theorem 2. 1] and Corollary 2. 2.
Assume (3). Let C be a cyclic left A-module, ${}_AE$ the injective hull of ${}_AC$. Suppose that $C\neq E$. If $b\in E$, $b\notin C$, let $B=Ab+C$, $D={}_AC\oplus{}_AB$, $i: C\to B$ the inclusion map, $j: C\to D$ and $k: B\to D$ the natural injections. Since ${}_AD$ is CF-injective, there exists a left A-homomorphism $h: D\to D$ such that $hki=j$. If $p: D\to C$ is the natural projection, then $phk: B\to C$ such that $(phk)i=p(hki)=pj=$identity map on C which proves that ${}_AC$ is a direct summand of ${}_AB$. But ${}_AC$ is essential in ${}_AB$ which yields $C=B$, contradicting $b\notin C$. This proves that $C=E$ is injective and (3) implies (4).
Since a finite direct sum of CF-injective left A-modules is CF-injective, then the above proof shows that (4) implies (5).
Finally, (5) implies (1) by [4, Theorem] and Proposition 3(1).

The proof of Theorem 7 yields the following sufficient conditions for CF-injective and MP-injective modules to be injective.

PROPOSITION 8. *The following conditions are equivalent*:
(1) *Every cyclic MP-injective left A-module is injective*;
(2) *For any cyclic MP-injective left A-module M, every essential extension of* $_AM$ *is CF-injective.*

PROPOSITION 9. *The following conditions are equivalent*:
(1) *Every CF-injective left A-module is injective*;
(2) *Every CF-injective left A-module is quasi-injective.*

PROPOSITION 10. *The following conditions are equivalent*:
(1) *A is left self-injective*;
(2) *Every essential extension of* $_AA$ *is CF-injective.*

Since an essential extension of a non-singular left A-module is non-singular and a direct sum of non-singular left A-modules is non-singular, [17, Corollary 6] and the proof of Theorem 7 lead to

THEOREM 11. *The following conditions are equivalent*:
(1) *A is left self-injective regular*;
(2) *For any principal left ideal P of A, every essential extension of* $_AP$ *is CF-injective*;
(3) *A is left non-singular such that every finitely generated non-singular left A-module is CF-injective.*

We conclude with the following question motivated by [5, Theorem 3.2 (3)].

QUESTION: Is A von Neumann regular if (1) every cyclic semi-simple left A-module is flat or (2) every cyclic semi-simple left A-module is *MP*-injective?

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